General and Differential Topology QR Exam – Jan 7, 2023

Problem 1. Let M be a smooth manifold, and let $f : M \to \mathbb{R}$ and $g = (g^1, \ldots, g^k) : M \to \mathbb{R}^k$ be smooth maps. Assume that 0 is a regular value of f, and let $X = f^{-1}(0) \subset M$. Suppose $p \in X$ is a point. Give, with proof, a condition on $df_p, dg^1_p, \ldots, dg^k_p$ that is equivalent to the point p being a critical point of $g|_X : X \to \mathbb{R}^k$.

Problem 2. Let M be the vector space of 2×2 real matrices. Let $X = SL(2) \subset M$ be the group of 2×2 real matrices of determinant 1. Let $G = \mathbb{R}^*$ be the multiplicative group of the reals. Let G act on M and X by

$$t \cdot A = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot A.$$

(a) Determine whether the orbit space M/G is Hausdorff.
(b) Determine whether the orbit space X/G is Hausdorff.

Problem 3. Consider the unit circle $S^1 = \{x^2 + y^2 = 1\} \subset \mathbb{R}^2$. Let $\iota : S^1 \to \mathbb{R}^2$ be the inclusion, giving S^1 the structure of a smooth submanifold.

(a) Describe a 1-form on S^1 that is closed but not exact.
(b) Using your result from the previous part and the fact that $H^1_{dR}(\mathbb{R}^2) = 0$, prove that there does not exist a smooth map $r : \mathbb{R}^2 \to S^1$ that is a retraction (i.e. $r \circ \iota = \mathsf{id}_{S^1}$).

Problem 4. Let M be a smooth manifold with smooth submanifold N. Let V be the vector space of vector fields v on M satisfying

$$v_p \in T_pN \text{ for all } p \in N.$$

Prove that V is closed under Lie bracket of vector fields.

Problem 5. Let G be the Lie group $GL(n)$ of invertible $n \times n$ real matrices. Let L be the subgroup of lower-triangular matrices with ones on the diagonal, and let U be the subgroup of upper-triangular matrices with ones on the diagonal. Consider the map

$$f : L \times U \to G$$

given by matrix multiplication: $f(A, B) = AB$. Prove that f is an injective immersion of smooth manifolds.