THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology

January 2023

1. Let Z_n be the quotient of the unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ by the equivalence relation generated by $z \sim e^{2\pi i/n} z$ when $|z| = 1$, with the quotient topology. For which values of $n = \{1, 2, 3, \ldots\}$ is Z_n a topological manifold without boundary (i.e. every point $u \in Z_n$ has a neighborhood homeomorphic to $\mathbb{R}^k$ for some k)?

2. Let a subrgoup G of the free group $F(a, b, c)$ on three elements a, b, c be generated by a^2, b^2, c^2 and the six words expressed by permutations of the letters a, b, c. Find a set of free generators of G. Is G a normal subgroup of $F(a, b, c)$?

3. A *wedge sum* of two non-empty spaces X, Y with specified points $x \in X$, $y \in Y$ is obained as the quotient of the disjoint union $X \amalg Y$ by the equivalence relation generated by $x \sim y$ (with quotient topology). Describe the universal cover of the wedge sum of S^2 and $S^1 \times S^1$.

4. Let $n \in \{0, 1, 2, \ldots\}$ and let X_n denote a CW-complex with 1 zero-cell, n one-cells and 1 two-cell. For what choices of n does there exist a covering map $X_n \to X_n$ which is not a homeomorphism

   (a) for at least one such X_n?

   (b) for all such X_n?

5. Let $Y = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$ and let $Z = \{(x, y, z, t) \in Y \mid z = t = 0\}$ (both with the subspace topology of $\mathbb{R}^4$). Calculate the homology of the quotient Y/Z.