THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology

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1. Let Z_n be the quotient of the unit disk $\{z \in \mathbb{C} \mid |z| \leq 1\}$ by the equivalence relation generated by $z \sim e^{2\pi i/n}z$ when $|z| = 1$, with the quotient topology. For which values of $n = \{1, 2, 3, \ldots\}$ is Z_n a topological manifold without boundary (i.e. every point $u \in Z_n$ has a neighborhood homeomorphic to $\mathbb{R}^k$ for some k)?

   Solution: The answer is for $n = 2$ only. For $n = 1$ we have D^2, which has boundary, for $n > 2$, at a point $|z| = 1$, Z_n is locally homeomorphic to the suspension Q of the wedge of n copies of $[0, 1)$ with base point 0. Thus, $H_2(Z_n, Z_n \smallsetminus \{z\}) \cong H_1(Q \smallsetminus \{(0, 0)\}) = \mathbb{Z}^{n-1}$, so it is not locally homeomorphic at z to a Euclidean space.

2. Let a subrgoup G of the free group $F(a, b, c)$ on three elements a, b, c be generated by a^2, b^2, c^2 and the six words expressed by permutations of the letters a, b, c. Find a set of free generators of G. Is G a normal subgroup of $F(a, b, c)$?

   Solution: The “casting out” technique for drawing the covering of $\bigvee_{\{a,b,c\}} S^1$ corresponing to G yields a complete graph on 4 vertices, with edges doubled up. Thus, G has 9 free generators. It is regular covering, thus the subgroup is normal (the quotient is $\mathbb{Z}/2 \times \mathbb{Z}/2$).

3. A *wedge sum* of two non-empty spaces X, Y with specified points $x \in X$, $y \in Y$ is obained as the quotient of the disjoint union $X \amalg Y$ by the equivalence relation generated by $x \sim y$ (with quotient topology). Describe the universal cover of the wedge sum of S^2 and $S^1 \times S^1$.

   Solution: The union of the $z = -1$ plane in $\mathbb{R}^3$ and shifts of the unit sphere by $\mathbb{Z}a \times \mathbb{Z}b \times \{0\}$ with $a, b > 2$ is one description.

4. Let $n \in \{0, 1, 2, \ldots\}$ and let X_n denote a CW-complex with 1 zero-cell, n one-cells and 1 two-cell. For what choices of n does there exist a covering map $X_n \to X_n$ which is not a homeomorphism

   (a) for at least one such X_n?

   (b) for all such X_n?

   Solution: Since X_n is compact, the covering must have finite degree, and thus the Euler characteristic of X_n must be 0. Therefore, $n = 2$ is a necessary conditon, in which case $S^1 \times S^1$ is an example. On the other hand, $X = ([0,1] \times [0,1])/(0,0) \sim (1,1)$ is a non-example, since $x = (0,0)$ is the only point at which X is not locally a 2-manifold with boundary (for example, using first local homology). However, a k-sheeted cover would have k such points, which is the contradiction. Thus, the answer to (a) is $n = 2$ only, and the answer to (b) is that no such n exists.

5. Let $Y = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}$ and let $Z = \{(x, y, z, t) \in Y \mid z = t = 0\}$ (both with the subspace topology of $\mathbb{R}^4$). Calculate the homology of the quotient Y/Z.

   Solution: The space Y is homotopy equivalent to $S^3 \vee S^2$, so $H_k(Y) = \mathbb{Z}$ for $k = 0, 2, 3$ and 0 otherwise.