General and Differential Topology QR Exam – May 2, 2022

with solutions

This exam consists of five problems. All manifolds are assumed to be C^∞. $\mathfrak{X}(M)$ denotes the space of smooth vector fields on the manifold M. All items will be graded independently of each other.

Problem 1. The subgroup Γ of $\mathrm{SU}(2)$ generated by the matrix

$$\gamma := \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SU}(2)$$

is isomorphic to $\mathbb{Z}_4$, and it acts on the unit three-sphere $S^3 \subset \mathbb{C}^2$ by matrix multiplication. Let $X = S^3/\Gamma$ be the orbit space, with the quotient topology. Answer the following questions, with proofs:

1. Is X is second countable?
2. Is X is Hausdorff?
3. Is the projection $\pi : S^3 \to X$ is a local homeomorphism?

Solution: (1)Yes. Since the action is continuous the projection π is an open map. Then, applying a standard result, X is second countable because S^3 is. (2) Yes. Similarly, a standard result is that since π is open X is Hausdorff iff the orbit relation is closed in $S^3 \times S^3$. Since the group is finite and S^3 compact, the orbit relation is the union of finitely-many closed sets (the graphs of the action of each group element), so it is closed. (3) Yes, because in addition the action is free (which should be checked).

Problem 2. A smooth $F : M \to M$ is called a *Lefshetz map* iff for all $p \in M$ such that $F(p) = p$ one has:

$$1 \in \mathbb{R} \text{ is not an eigenvalue of } \quad dF_p : T_pM \to T_pM.$$

1. Show that F is Lefschetz iff its graph and the diagonal $\Delta \subset M \times M$ intersect transversely.
2. Show that if F is Lefschetz then the set $\{p \in M : F(p) = p)\}$ consists of isolated points.
3. Is the converse of the previous statement true?

Solution: Let $\Gamma \subset M \times M$ be the graph of F; this is a regular submanifold because it is the image of the proper embedding $M \ni p \mapsto (p, F(p)) \in M \times M$. Let $(p, p) \in \Gamma \cap \Delta$, i.e. $F(p) = p$. Then

$$T_{(p,p)}\Gamma = \{(v, dF_p(v)) \mid v \in T_pM\} \quad \text{and} \quad T_{(p,p)}\Delta = \{(v, v) \mid v \in T_pM\}$$

Both of these spaces have dimension n, the dimenson of M, so the intersection is transversal iff their intersection is zero, that is iff $\forall v \in T_pM \ dF_p(v) = v \Rightarrow v = 0$.

(2) By transversality, the intersection $\Gamma \cap \Delta$ is a manifold of codimension $2n$, that is, it has dimension zero. Therefore it consists of isolated points.

(3) No, it suffices to consider a map $F : (0,1) \to (0,1)$ with finitely-many fixed points whose graph is tangent to the diagonal at one point.

Problem 3. Let $S^2 \subset \mathbb{R}^3$ be the two-sphere with the standard orientation, and $F : S^2 \to S^2$ be given by $F(a,b,c) = (a,-b,-c)$. Note that F is a diffeomorphism (no proof needed). Also, let x, y, z denote the restrictions of the coordinate functions to S^2, and let $\alpha = xydy \wedge dz$.

1. Establish whether or not F is orientation preserving, and compute $F^*(\alpha)$. What do your findings imply about the value of $\int_{S^2} \alpha$? Explain.

2. The vector field $\langle -y, x, 0 \rangle$ in $\mathbb{R}^3$ is tangent to the sphere, and therefore it restricts to a vector field $X \in \mathfrak{X}(S^2)$. Compute the Lie derivative $\mathcal{L}_X \alpha$.

Solution: (1) Since the sphere is connected, it suffices to check whether dF_p is orientation preserving or not at a single $p \in S^2$. The simplest choice is $p = (1,0,0)$, so the tangent space is identified with the yz plane. It is clear that dF_p is multiplication by (-1), which is orientation-preserving in dimension two. Moreover, since $F^*x = x$, $F^*y = -y$ and $F^*z = -z$, one has $F^*\alpha = x(-y)d(-y) \wedge d(-z) = -\alpha$. Since integration is invariant under pull-backs by orientation-perserving diffeomorphisms, $\int_{S^2} \alpha = \int_{S^2} F^*\alpha = -\int_{S^2} \alpha$, and therefore $\int_{S^2} \alpha = 0$.

(2) We apply Cartan's magic formula. Note that $d\alpha = 0$ by dimensional considerations, so we are left with $\mathcal{L}_X(\alpha) = d\iota_X \alpha = d(x^2 ydz) = 2xydx \wedge dz + x^2 dy \wedge dz$.

Problem 4. Let M be a smooth manifold, and $X \in \mathfrak{X}(M \times \mathbb{R})$ be a smooth vector field of the form

$$\forall (p,s) \in M \times \mathbb{R} \qquad X_{(p,s)} = (V_{(p,s)}, \partial_s), \qquad \text{where} \quad V_{p,s} \in T_pM.$$

(We are identifying $T_{(p,s)}(M \times \mathbb{R})$ with $T_pM \times T_s\mathbb{R}$.) For each $(p,s) \in M \times \mathbb{R}$, let $t \mapsto \Phi_t(p,s)$ be the integral curve of X starting at (p,s), and denote

$$\phi_{t,s}(p) := \pi\left(\Phi_{t-s}(p,s)\right), \quad \text{where} \quad \pi : M \times \mathbb{R} \to M \quad \text{is the projection.}$$

1. Show that $\forall t_0 \in \mathbb{R}$, $p \in M$ the curve on M $t \mapsto \gamma(t) = \phi_{t,t_0}(p)$ is defined in a neighborhood of t_0 and satisfies $\dot\gamma(t) = V_{\gamma(t),t}$, $\gamma(t_0) = p$.

2. Assuming that X is complete, show that $\forall r, s, t \in \mathbb{R}$

$$\phi_{t,s} \circ \phi_{s,r} = \phi_{t,r},$$

where $\phi_{t,s} : M \to M$ is the map $p \to \phi_{t,s}(p)$, etc.

Solution: (1) The integral curve of X, $u \mapsto \Phi_u(p,s)$, is defined for u in a neighborhood of zero. (2) By definition, $\Phi_{t-s}(p,s) = (\phi_{t,s}(p), t)$, or $\Phi_u(p,s) = (\phi_{u+s,s}(p), u+s)$.

Using that Φ is a one-parameter group,

$$\Phi_t \circ \Phi_u(p,s) = \Phi_t(\phi_{u+s,s}(p), u+s) = (\phi_{t+u+s,u+s} \circ \phi_{u+s,s}(p), u+s+t)$$
$$= \Phi_{t+u}(p,s) = (\phi_{t+u+s,s}(p), t+u+s),$$

or

$$\phi_{t+u+s,s}(p) = \phi_{t+u+s,u+s} \circ \phi_{u+s,s}(p).$$

A change of variables yields the desired result.

Problem 5. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_eG$ (where e is the identity), and let $A, B \in \mathfrak{g}$ be linearly independent elements satisfying $[A, B] = 0$.

1. Carefully explain why $\forall s, t \in \mathbb{R}$ $\exp(sA)\exp(tB) = \exp(tB)\exp(sA)$, where $\exp : \mathfrak{g} \to G$ is the exponential map.
2. Show that the map $E : \mathbb{R}^2 \ni (s,t) \mapsto \exp(sA)\exp(tB) \in G$ is a an immersion.
3. Specialize to the case $G = \mathrm{U}(3)$, and A, B diagonal with diagonal entries the components of $i\vec{\lambda} = \langle i\lambda_1, i\lambda_2, i\lambda_3\rangle$ and $i\vec{\mu} = \langle i\mu_1, i\mu_2, i\mu_3\rangle$ respectively.

   Under what conditions on $\vec{\lambda}, \vec{\mu} \in \mathbb{R}^3$ is the image of the map E a closed (regular) submanifold of $\mathrm{U}(3)$?

Solution: (1) Denote by $A^\sharp$, $B^\sharp$ the left-invariant vector fields corresponding to A and B. By left-invariance, left multiplication by any $g \in G$ maps integral curves of $B^\sharp$ to integral curves, and therefore

$$\exp(sA)\exp(tB) = \phi_t^{B^\sharp}(\exp(sA)) = \phi_t^{B^\sharp} \circ \phi_s^{A^\sharp}(e)$$

since $\exp(sA) = \phi_s^{A^\sharp}(e)$. A similar expression holds with sA and tB exchanged. On the other hand, the assumption on A and B is that $A^\sharp$, $B^\sharp$ commute as vector fields. Therefore, their flows $\phi^{A^\sharp}$ and $\phi^{B^\sharp}$ commute, and the result follows.

(2) Pick $(s,t) \in \mathbb{R}^2$ and compute

$$dE_{(s,t)}(\partial_t) = dL_{\exp(sA)}(B^\sharp_{\exp(tB)}) = B^\sharp_{E(s,t)} = dL_{E(s,t)}(B)$$

where $L_g : G \to G$ is left translation by $g \in G$. Similarly, $dE_{(s,t)}(\partial_s) = dL_{E(s,t)}(A)$. Since A, B are lineraly independent and $dL_{E(s,t)} : T_eG \to T_{E(s,t)}G$ is an isomorphism, the result follows.

(3) One computes that $E(s,t)$ is the diagonal matrix with entries $e^{i(s\lambda_j + t\mu_j)}$, $j = 1, 2, 3$. Observe that the image of E is contained in the 3-torus $T \subset \mathrm{U}(3)$ of diagonal matrices, and the question is equivalent to whether the image of E is a submamifold of T. If $\Pi \subset \mathbb{R}^3$ is the span of $\{\vec{\lambda}, \vec{\mu}\}$, the condition is that

$$\Pi \cap 2\pi\mathbb{Z}^3$$

be a lattice in Π.