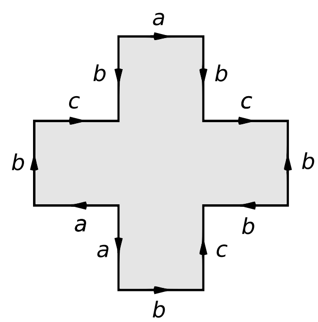
Algebraic Topology QR Exam – May 2022

1. Let X be the quotient space defined by the following polygon with edge identifications. Compute $\pi_1(X)$.



2. Let X be the wedge of 2 circles a and b with single vertex v. By mild abuse of notation we write a and b to mean both the edges of the graph X and the corresponding generators of $\pi_1(X, v)$. Consider the following two covers $p : \tilde{X} \to X$, with the map p specified by the edge labels and orientations of $\tilde{X}$. A distinguished lift $\tilde{v}$ of v is marked with a gray dot. Let $H = p_*(\pi_1(\tilde{X}, \tilde{v}))$. For each cover, state with (very) brief justification,

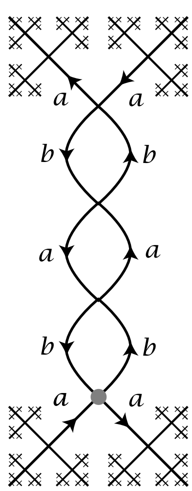
   (i) a free generating set for H,

   (ii) the index of H as a subgroup of $\pi_1(X, v)$,

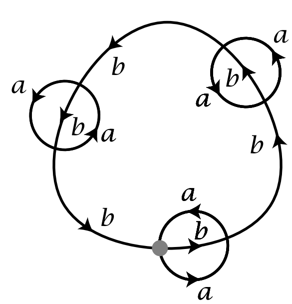
   (iii) whether the cover is regular,

   (iv) the deck group of the cover (as an abstract group),

   (v) generators for the normalizer of H in $\pi_1(X, v)$.



(a)



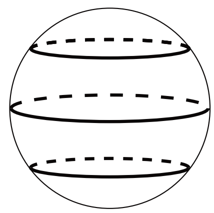
(b)

3. Let $p : \tilde{X} \to X$ be a covering map of path-connected spaces, and let $\tilde{x_0} \in \tilde{X}$. Let $x_0 = p(\tilde{x_0})$. For each of the following statements: either prove the statement using the definition and/or lifting properties of a covering space, or construct a counterexample. In a proof, give complete theorem statements for any lifting properties you cite.

   (i) The induced map $p_* : \pi_1(\tilde{X}, \tilde{x_0}) \to \pi_1(X, x_0)$ is injective.

   (ii) The induced map $p_* : H_1(\tilde{X}) \to H_1(X)$ is injective.

4. Let N be a positive integer. Let X be the 2-sphere, and let $A \subseteq X$ be the union of N disjoint circles of latitude (pictured below for $N = 3$). Let X/A be the quotient space with A collapsed to a point. Compute $\widetilde{H}_*(X/A)$.



5. Let M be an n-manifold for some $n \geq 1$, and let $x \in M$. Consider the pair $(X, A) = (M, M \backslash \{x\})$.

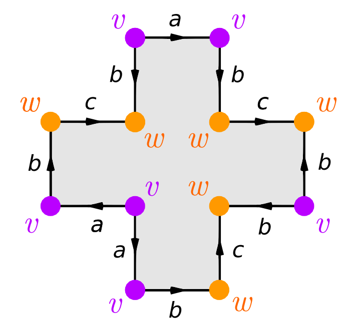
   (a) Consider the quotient space X/A where A is collapsed to a point. Describe the topology on X/A and show (by writing an explicit homotopy and verifying continuity) that this space is contractible.

   (b) Prove that $H_*(X, A)$ and $\widetilde{H}_*(X/A)$ are not equal.

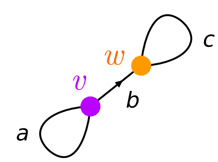
Solutions

1. The description of X as a polygon with edge identifications naturally suggests a CW complex structure on X. Let X^1 denote the 1-skeleton of X, with edges a, b, c. The fundamental group of a CW complex X is the quotient of the (necessarily free) group $\pi_1(X^1)$ by a relator for each 2-cell in the 2-skeleton of X.

   By tracing through the edge identifications we find that X has two distinct vertices v and w.

   Thus its 1-skeleton X^1 is as shown.

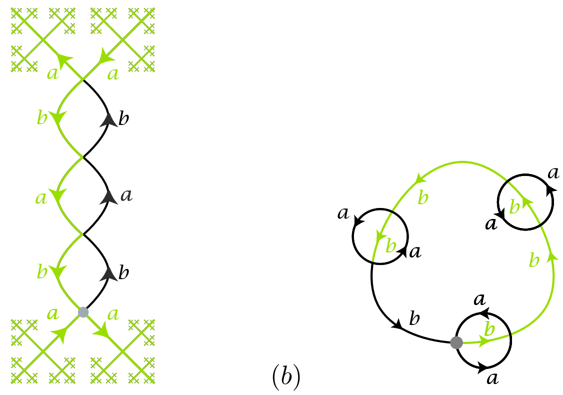
   

   Take v to be the basepoint. The edge b is a maximal tree in the 1-skeleton. Hence $\pi_1(X^1, v)$ is the free group on the two generators $A = [a]$ and $C = [bcb^{-1}]$.

   The CW complex X has a single 2-cell. It is glued, starting in the top left corner, along the word (read left-to-right)
   $$abcb^{-1}bc^{-1}b^{-1}a^{-1}abcb^{-1}.$$
   As an element of $\pi_1(X^1, v)$, this word is equivalent to the based loop
   $$\begin{aligned}&[abcb^{-1}bc^{-1}b^{-1}a^{-1}abcb^{-1}]\\ &= [abcc^{-1}b^{-1}bcb^{-1}]\\ &= [abcb^{-1}]\\ &= AC\end{aligned}$$
   Thus $\pi_1(X) \cong \langle\, A, C \mid AC \,\rangle$. This relation implies that $C = A^{-1}$, so we can simplify the presentation $\pi_1(X) \cong \langle\, A \mid \,\rangle \cong \mathbb{Z}$.

   Remark: The space X is not a surface: it is not locally Euclidean; each boundary segmented is glued to three or more other segments. We therefore cannot use Euler characteristic / classification of surfaces arguments to compute $\pi_1(X)$.

2. (i) To find a free generating set of a graph, we must make a choice of maximal tree T (i.e. a contractible subgraph containing every vertex). For example, we could choose the trees highlighted in green below.



(a) (b)

Then $\pi_1(\tilde{X})$ is generated by a free generating set corresponding to a loop for every edge in the complement of the chosen tree T. For each such edge $e \in \tilde{X} \setminus T$, we find a loop based at $\tilde{v}$ that passes through the tree T, traverses e once, and returns to $\tilde{v}$ through T.

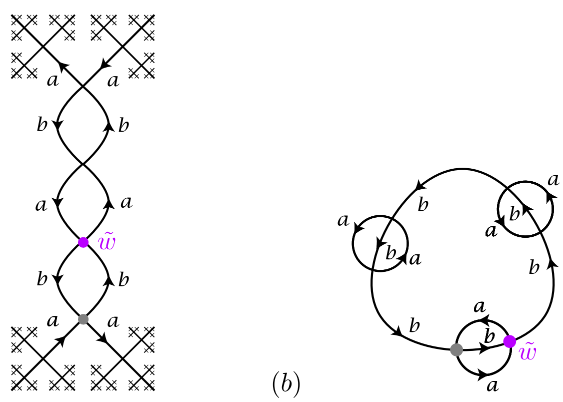
The induced map p_* on π_1 is injective, so we must determine the image of each free generator under p. With these choices of maximal trees we obtain,

(a) H is freely generated by b^2, ba^2b, bab^2ab.
(b) H is freely generated by $ba, ba^{-1}, b^3ab^{-2}, b^3a^{-1}b^{-2}, b^5ab^4, b^5a^{-1}b^4, b^6$.

(ii) The index of H is equal to the degree of the cover p. This is equal to the cardinality of $p^{-1}(v)$, the number of vertices of the graph $\tilde{X}$.

(a) The index is (countably) infinite.
(b) The index is 6.

(iii) We can show in both cases that the covers are not regular, as the group of deck maps does not act transitively on the set of vertices $p^{-1}(v)$ of $\tilde{X}$. Consider the vertices $\tilde{w}$ labelled below.



(a) (b)

(a) By inspection, there is no deck map (in fact no graph automorphism) mapping $\tilde{v}$ to $\tilde{w}$.
(b) The only graph automorphisms mapping $\tilde{v}$ to $\tilde{w}$ reverse the orientation of the edge $[\tilde{v}, \tilde{w}]$, and hence are not valid deck maps.

(iv) The deck groups are the subgroup of graph automorphisms of the covers $\tilde{X}$ that preserve the labels and orientations of the edges. We compute them by studying (visually) the symmetry of the graphs. (In principle we could alternatively compute them by calculating the normalizers $N(H)$ and the quotient groups $N(H)/H$ algebraically).

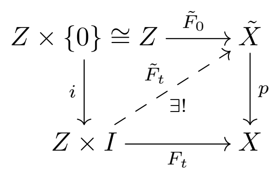
(a) By inspection, the only nontrivial deck map is 180° rotation in the plane of the page. Hence the deck group is $\mathbb{Z}/2\mathbb{Z}$.
(b) By inspection, the deck group is generated by 120° rotation in the plane of the page. Thus it is $\mathbb{Z}/3\mathbb{Z}$.

(v) A loop in $\pi_1(X, v)$ is in the normalizer $N(H)$ if and only if its lift to $\tilde{X}$ starting at $\tilde{v}$ is a path from $\tilde{v}$ to a vertex $\tilde{w}$ such that $\tilde{w}$ is in the orbit of $\tilde{v}$ under the deck group. (See for example the proof of Proposition 1.39 in Hatcher). For any given vertex $\tilde{w} \in p^{-1}(v)$, the set of loops in $\pi_1(X, v)$ that lift to a path from $\tilde{v}$ to $\tilde{w}$ comprise a right coset of the subgroup H in $\pi_1(X, v)$. It follows that the normalizer $N(H)$ of H is generated by H and, for each vertex $\tilde{w}$ in the deck group orbit of $\tilde{v}$, a choice of loop $\gamma \in \pi_1(X, v)$ that lifts to a path from $\tilde{v}$ to $\tilde{w}$.

(a) The normalizer is generated by H and the element bab.
(b) The normalizer is generated by H, b^2, and b^4.

3. (a) This statement is true (and is foundational to the classification theorem of covering spaces). Recall that covering maps satisfy the following homotopy lifting property.

Theorem (Covering maps have the homotopy lifting property). Let $p : \tilde{X} \to X$ be a covering map, and let $F_t : Z \times I \to X$ be a homotopy of maps $Z \to X$. Then given any lift $\tilde{F}_0 : Z \to \tilde{X}$ of F_0, there exists a unique lift $\tilde{F}_t : Z \times I \to \tilde{X}$ of F_t whose restriction to $t = 0$ is the lift $\tilde{F}_0$.



Before we proceed, we will use the homotopy lifting property to deduce the following lemma.

Lemma (Lifts of constant paths are constant). Let $p : \tilde{X} \to X$ be a covering map. Let $\alpha : I \to X$ be the constant path at some point $y_0 \in X$. Then any lift $\tilde{\alpha} : I \to \tilde{X}$ of α to $\tilde{X}$ is a constant path.

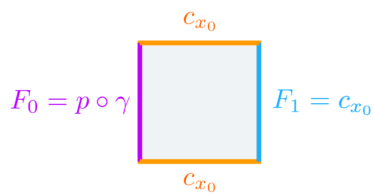
Consider such a constant path $\alpha : I \to X$. We may view α as a homotopy of maps from a point to X. A choice of lift $0 \mapsto \tilde{y_0}$ of the map $\alpha|_{\{0\}} : 0 \mapsto y_0$ is equivalent to a choice of preimage $\tilde{y_0} \in p^{-1}(y_0)$. Given a choice $\tilde{y_0}$ we may lift α to the constant path $\tilde{\alpha}$ at $\tilde{y_0}$. But then the uniqueness condition in the homotopy lifting property implies that these constant paths are the only lifts of α.

With this homotopy lifting property and this lemma, we will prove that the induced map p_* on π_1 is injective.

Suppose we have an element $[\gamma] \in \pi_1(\tilde{X}, \tilde{x_0})$ such that $p_*([\gamma]) := [p \circ \gamma]$ is trivial in $\pi_1(X, x_0)$. Our goal is to show that $[\gamma]$ is trivial in $\pi_1(\tilde{X}, \tilde{x_0})$. Since $p_*([\gamma]) = 1$, there exists a homotopy $F_t : I \times I \to X$ of based loops from $p \circ \gamma$ to the constant loop c_{x_0} at x_0. Concretely, this is a homotopy F_t satisfying

- $F_0 = p \circ \gamma$
- $F_1 = c_{x_0}$
- $F_t(0) = F_t(1) = x_0$ for all t.

The domain $I \times I$ of F is illustrated below.



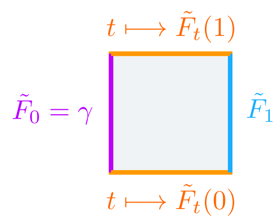
Since γ is a lift of $F_0 = p \circ \gamma$, we can apply our homotopy lifting theorem to obtain a lift $\tilde{F}_t : I \times I \to \tilde{X}$ of F extending γ,

$$\begin{array}{ccc} I \times \{0\} \cong I & \xrightarrow{\tilde{F}_0 = \gamma} & \tilde{X} \\ {\scriptstyle i}\downarrow & \overset{\tilde{F}_t}{\nearrow} \; \exists! & \downarrow{\scriptstyle p} \\ I \times I & \xrightarrow[F_t]{} & X \end{array}$$

To complete the problem, we will verify that $\tilde{F}_t$ is a based homotopy of loops from γ to the constant loop at $\tilde{x_0}$. Concretely, we must check

- $\tilde{F}_0 = \gamma$
- $\tilde{F}_1 = c_{\tilde{x_0}}$
- $\tilde{F}_t(0) = \tilde{F}_t(1) = \tilde{x_0}$ for all t.

The domain $I \times I$ of $\tilde{F}$ is illustrated below.



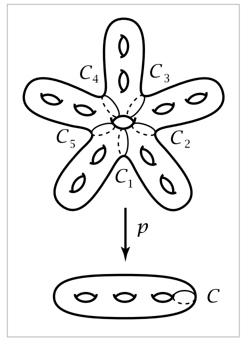
The property $\tilde{F}_0 = \gamma$ follows from our construction of $\tilde{F}$. Consider the restriction of $\tilde{F}$ to the maps

$$\begin{aligned} I &\longrightarrow \tilde{X} \\ s &\longmapsto \tilde{F}_1(s) \end{aligned} \qquad \begin{aligned} I &\longrightarrow \tilde{X} \\ t &\longmapsto \tilde{F}_t(0) \end{aligned} \qquad \begin{aligned} I &\longrightarrow \tilde{X} \\ t &\longmapsto \tilde{F}_t(1) \end{aligned}$$

These are lifts of the constant paths F_1, $t \mapsto F_t(0)$, and $t \mapsto F_t(1)$. By the lemma, these lifts must be constant paths. Since γ is a based loop at $\tilde{x_0}$, we see $\tilde{F}_0(0) = \tilde{F}_0(1) = \tilde{x_0}$. We conclude the second and third paths (hence also the first path) must be the constant paths at $\tilde{x_0}$. This concludes the proof.

(b) This statement is false, and many common examples of path-connected covering spaces $p : \tilde{X} \to X$ with nonabelian fundamental groups will be counterexamples.

For example, consider any cover $p : \Sigma_h \to \Sigma_g$ (of degree $d > 1$) of closed orientable surfaces with $g > 1$, such as the following (image from Hatcher p82).

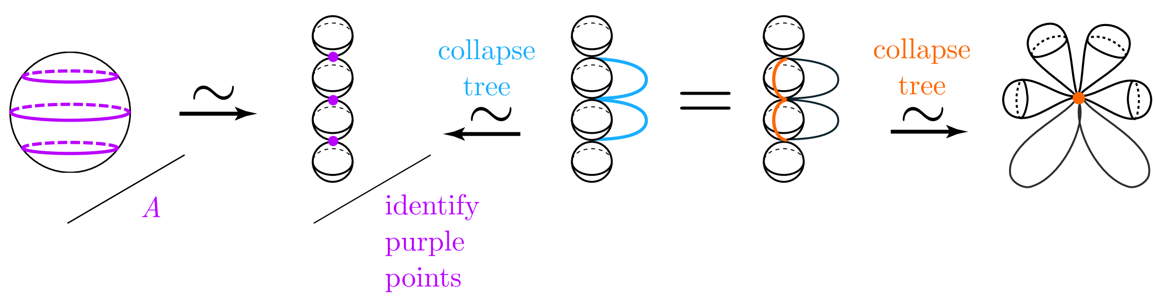


Then by an Euler characteristic argument, $2-2h = d(2-2g)$, so $h = d(g-1)+1 > g$. But then $H_1(\Sigma_h) \cong \mathbb{Z}^{2h}$ does not inject into $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$.

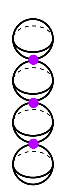
Alternatively, consider any degree-d cover $p : \tilde{X} \to X$ of a wedge of two circles X by a path-connected graph $\tilde{X}$. Again an Euler characteristic argument implies that $\tilde{X}$ is homotopy equivalent to a wedge of $(d+1)$ circles, but $H_1(\tilde{X}) \cong \mathbb{Z}^{d+1}$ does not embed into $H_1(\tilde{X}) \cong \mathbb{Z}^2$ for any $d > 1$. In particular, both covers from Problem 2 are counterexamples. More generally, any positive degree cover of path-connected finite graphs is a counterexample.

4. *Approach #1 : Explicit homotopy equivalence.* The space X/A is homotopy equivalent to a wedge of $(N+1)$ 2-spheres and $(N-1)$ circles. We will exhibit an explicit homotopy equivalence (illustrated for $N = 3$) by repeatedly applying the following principle: Given a CW complex Y and a contractible CW subcomplex S, the quotient map $Y \to Y/S$ is a homotopy equivalence.

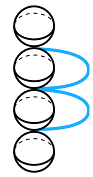
   Our argument is summarized in this figure:



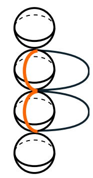
   We first collapse each of the N circles of latitude to a point. The result is $(N+1)$ 2-spheres, each wedged to its neighboring sphere(s) at one of N distinguished points, as shown. Call this space Z.



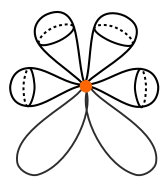
   The space X/A is the quotient of Z identifying these N wedge points. Consider the space W obtained by gluing $(N-1)$ edges to Z as shown.



Since the $(N-1)$ edges form a contractible subcomplex, the quotient map collapsing them to a point is a homotopy equivalence. Thus X/A is homotopy equivalent to W. On the other hand, consider the contractible subcomplex of W consisting of the $(N-1)$ edges shown below.



By collapsing this subcomplex, we see that W is homotopy equivalent to a wedge of $(N-1)$ circles and $(N+1)$ 2-spheres.



Thus

$$X/A \simeq \left(\bigvee^{N+1} S^2 \right) \vee \left(\bigvee^{N-1} S^1 \right).$$

We know for all n and spaces A, B,

$$\widetilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, \ k = n \\ 0, \ \text{otherwise}, \end{cases} \qquad H_k(A \vee B) \cong H_k(A) \oplus H_k(B).$$

We further know X/A is path-connected so $\widetilde{H}_0(X/A) = 0$. We conclude,

$$\widetilde{H}_k(X/A) \cong \begin{cases} 0, \ k = 0 \\ \mathbb{Z}^{N-1}, \ k = 1 \\ \mathbb{Z}^{N+1}, k = 2 \\ 0, \ \text{otherwise.} \end{cases}$$

Approach #2 : The long exact sequence of a pair. Since we can realize X as a CW complex with A a CW subcomplex, the pair (X, A) is a good pair, and $H_*(X, A) \cong \widetilde{H}_*(X/A)$. Thus we can study these groups using the long exact sequence of a pair.

We know that X/A is path-connected (being the continuous image of $X = S^2$ under a quotient map), so $\widetilde{H}_0(X/A) = 0$. Moreover, it is a 2-dimensional CW complex, so $\widetilde{H}_k(X/A) = 0$ for $k > 2$. It remains to compute $\widetilde{H}_k(X/A)$ for $k = 1, 2$.

Since $X = S^2$ and $A \cong \sqcup_N S^1$, we find

$$\widetilde{H}_k(X) \cong \widetilde{H}_k(S^2) \cong \begin{cases} \mathbb{Z}, \ k = 2 \\ 0, \ \text{otherwise.} \end{cases}$$

$$\widetilde{H}_k(A) \cong \widetilde{H}_k(\sqcup_N S^1) \cong \begin{cases} \mathbb{Z}^{N-1}, \ k=0 \\ \mathbb{Z}^N, \ k=1 \\ 0, \ \text{otherwise.} \end{cases}$$

Then the long exact sequence of pair is as follows.

$$\begin{array}{ccccccccc} & & & & \cdots & \longrightarrow & \widetilde{H}_3(X) & \longrightarrow & \widetilde{H}_2(A) \\ & & & & & & \| & & \| \\ & & & & & & 0 & & 0 \\ \longrightarrow & \widetilde{H}_2(X) & \longrightarrow & \widetilde{H}_2(X/A) & \longrightarrow & & \widetilde{H}_1(A) & \longrightarrow & \widetilde{H}_1(X) \\ & \| & & \| & & & \| & & \| \\ & \mathbb{Z} & & \widetilde{H}_2(X/A) & & & \mathbb{Z}^N & & 0 \\ \longrightarrow & \widetilde{H}_1(X/A) & \longrightarrow & \widetilde{H}_0(A) & \longrightarrow & & \widetilde{H}_0(X) & \longrightarrow & \widetilde{H}_0(X/A) \longrightarrow 0. \\ & \| & & \| & & & \| & & \| \\ & \widetilde{H}_1(X/A) & & \mathbb{Z}^{N-1} & & & 0 & & 0 \end{array}$$

The groups $\widetilde{H}_1(X/A)$ and $\widetilde{H}_2(X/A)$ sit in exact sequences of abelian groups

$$0 \longrightarrow \widetilde{H}_1(X/A) \longrightarrow \mathbb{Z}^{N-1} \longrightarrow 0$$

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{H}_2(X/A) \longrightarrow \mathbb{Z}^N \longrightarrow 0$$

Exactness of the first sequence implies $\widetilde{H}_1(X/A) \cong \mathbb{Z}^{N-1}$.

Consider the second sequence. A standard result from abstract algebra states that, if ever the quotient group in a short exact sequence of abelian groups is free abelian, then the short exact sequence splits. It follows that $\widetilde{H}_2(X/A) \cong \mathbb{Z}^{N+1}$. We will also verify this result directly without quoting this algebra fact.

Since X/A has the structure of a finite CW complex, the group $\widetilde{H}_2(X/A)$ is a finitely generated abelian group. Suppose $\widetilde{H}_2(X/A)$ contained a torsion element t. Since the quotient group $\mathbb{Z}^{N-1}$ is free abelian, the element t must be contained in the kernel of the map $\widetilde{H}_2(X/A) \to \mathbb{Z}^N$. But the kernel of this map is isomorphic to $\mathbb{Z}$, hence $t = 0$. Thus $\widetilde{H}_2(X/A)$ is free abelian. (We could also infer this result directly, as X/A has no 3-cells). Then, we apply the rank-nullity theorem (for $\mathbb{Z}$-linear maps) to the map $\widetilde{H}_2(X/A) \to \mathbb{Z}^N$ and we deduce that $\widetilde{H}_2(X/A) \cong \mathbb{Z}^{N+1}$.

Again we conclude

$$\widetilde{H}_k(X/A) \cong \begin{cases} 0, \ k=0 \\ \mathbb{Z}^{N-1}, \ k=1 \\ \mathbb{Z}^{N+1}, k=2 \\ 0, \ \text{otherwise.} \end{cases}$$

5. (a) Let $q : M \to M/(M \setminus \{x\})$ be the quotient map. Recall that, by definition of the quotient topology, a set U in the quotient space is open if and only if $q^{-1}(U)$ is open. The quotient space $M/(M \setminus \{x\})$ has, as a set, two points: the image $\overline{x}$ of x, and the image $\overline{y}$ of $M \setminus \{x\}$. Manifolds have the T_1 property (points are closed), so $q^{-1}(y) = M \setminus \{x\}$ is open. Thus $\{\overline{y}\}$ is open in the quotient. Since $\dim(M) > 0$, the point $q^{-1}(\overline{x}) = \{x\}$ is not open in M, so $\{\overline{x}\}$ is not open in the quotient. The quotient is the therefore the set $\{\overline{x}, \overline{y}\}$ with the following topology,

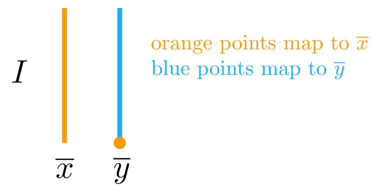
$$\{\overline{x}, \overline{y}\}, \{\overline{y}\}, \varnothing,$$

sometimes called *Sierpiński space*. To show this space is contractible, we will write an explicit homotopy $F : I \times \{\overline{x}, \overline{y}\} \to \{\overline{x}, \overline{y}\}$. There are many possible such homotopies; here is one example.

$$F(t,z) = \begin{cases} \overline{x}, & \text{if } z = \overline{x} \text{ and } t \in [0,1], \text{ or } z = \overline{y} \text{ and } t = 1 \\ \overline{y}, & \text{if } z = \overline{y} \text{ and } t \in [0,1) \end{cases}$$

.

This map F is illustrated below.



$I \times \{\overline{x}, \overline{y}\}$

Then F_0 is the identity map on $\{\overline{x}, \overline{y}\}$, and F_1 is the constant map at the point $\overline{x}$. Moreover, since the domain of F only has one proper nonempty open subset $\{\overline{y}\}$, to verify that F is continuous, we only need to verify that the preimage of $\{\overline{y}\}$ is open. But $F^{-1}(\overline{y}) = [0,1) \times \{\overline{y}\}$ (shaded blue in the figure) and this subset is open in the product topology on $I \times \{\overline{x}, \overline{y}\}$. Thus F_t is a contraction of the quotient space to the point $\overline{x}$, and we conclude that $M/(M \setminus \{x\})$ is contractible.

(b) Let $U \cong \mathbb{R}^n$ be a small Euclidean ball about the point x. By excision, $H_n(M, M \setminus \{x\}) \cong H_n(U, U \setminus \{x\})$. Since U is contractible and $U \setminus \{x\} \simeq S^{n-1}$, the long exact sequence of a pair

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \widetilde{H}_n(U) & \longrightarrow & H_n(U, U \setminus \{x\}) & \longrightarrow & \widetilde{H}_{n-1}(U \setminus \{x\}) & \longrightarrow & \widetilde{H}_{n-1}(U) & \longrightarrow \cdots \\ & \| & & \| & & \| & & \| & \\ & \widetilde{H}_n(\mathbb{R}^n) & & \widetilde{H}_n(U, U \setminus \{x\}) & & \widetilde{H}_{n-1}(S^{n-1}) & & \widetilde{H}_n(\mathbb{R}^n) & \\ & \| & & \| & & \| & & \| & \\ & 0 & & \widetilde{H}_n(U, U \setminus \{x\}) & & \mathbb{Z} & & 0 & \end{array}$$

implies that $H_n(U, U \setminus \{x\}) \cong \mathbb{Z}$ for any $n \geq 1$. But by part (a) the quotient $M/(M \setminus \{x\})$ is contractible and hence $\widetilde{H}_n(M/(M \setminus \{x\})) = 0$.