General and Differential Topology QR Exam – Aug 17, 2022

Problem 1. Let M be a smooth manifold.

(a) Prove that the total space of the tangent bundle TM is orientable.

(b) Now suppose M is the union of two orientable open submanifolds U_1, U_2 and the intersection $U_1 \cap U_2$ is connected. Prove that M is orientable.

Problem 2. Let $M = \mathbb{R}^2/\mathbb{Z}^2$ $(= S^1 \times S^1)$. Let $\iota : M \to M$ be the involution $\iota(x, y) = (-x, y+0.5)$. Let Q be the quotient of M by ι, with a natural smooth manifold structure inherited from $\mathbb{R}^2$.

(a) Explain how to interpret dy as defining a 1-form on Q.

(b) Show that, viewed as a 1-form on Q, dy is closed but not exact (so gives a nonzero element in $H^1_{\mathrm{dR}}(Q)$).

Problem 3. Let $\mathrm{SL}(n)$ be the group of $n \times n$ real matrices with determinant 1, considered as a submanifold of the vector space $\mathcal{M}(n)$ of all $n \times n$ real matrices. For each $g \in \mathrm{SL}(n)$ identify the tangent space $T_g \,\mathrm{SL}(n)$ with a linear subspace of $\mathcal{M}(n)$.

(a) Explicitly compute the linear subspace $T_g \,\mathrm{SL}(n)$ for an arbitrary $g \in \mathrm{SL}(n)$.

(b) Let $F : \mathrm{SL}(n) \to \mathrm{SL}(n)$ be the map $F(g) = gg^T$, where g^T is the transpose of the matrix g. Explicitly compute the map on tangent spaces $dF_{\mathsf{id}} : T_{\mathsf{id}} \,\mathrm{SL}(n) \to T_{\mathsf{id}} \,\mathrm{SL}(n)$, where id is the identity matrix. What is the rank of dF_{id}?

Problem 4. Here are two unrelated questions about submersions of smooth manifolds $\pi : M \to B$.

(a) Suppose that $\dim M = \dim B$ and M is compact. Prove that $\pi^{-1}(q)$ is a finite set for any point $q \in B$.

(b) Suppose that Y is a smooth vector field on B. Prove that there exists a smooth vector field X on M such that X is π-related to Y (i.e. $\pi_*(X(p)) = Y(\pi(p))$ for any point $p \in M$). (Hint: first show that you can find such an X locally on M, then use a partition of unity argument.)

Problem 5. Let M be a compact smooth manifold. Let X be a smooth vector field on M. For each part, just give a brief explanation or brief description of a counterexample.

(a) Is X necessarily complete? (A complete vector field is one such that all integral curves extend to be defined for all $t \in \mathbb{R}$.)

(b) Now assume that X is complete, and take the integral curve through some point p, $\phi_p : \mathbb{R} \to M$. Is ϕ_p necessarily an immersion?

(c) Now assume an integral curve ϕ_p is an immersion. Is the image of ϕ_p necessarily closed in M?