THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

**Qualifying Review examination in AlgebraicTopology**
Solutions

August 2022

1. Consider the set $Z \subset \mathbb{R}^3$ of all triples $(x_1, x_2, x_3)$ where for some $i \in \{1, 2, 3\}$ we have $|x_i| \leq 1$ and $|x_j| = 1$ for all $j \neq i$. Compute $\pi_1(\mathbb{R}^3 \smallsetminus Z)$.

   Solution: $Z$ is the union of the edges of a cube. Homotopy equivalently, we can draw the edges a graph in the plane, and attach to $S^2$ a copy of $S^1$ for each face. There are 5 faces, so the answer is the free group on 5 generators.

2. Consider the set $T$ of all $2 \times 2$ diagonalizable real matrices with determinant 0, with the subspace topology in $\mathbb{R}^4$. Is $T$ a topological manifold? Prove or disprove.

   Solution: $T$ is not a topological manifold. $T$ is contractible (with the contraction given by scaling), and hence connected. Removing the 0 matrix, we get the subset $T_0$ of matrices of rank 1. This space is disconnected, since it has a non-constant locally constant function given by the sign of the non-zero eigenvalue. A connected manifold which after removing a point becomes disconnnected has to be of dimension 1. However, we see that $T_0$ is a manifold of dimension $> 1$.

3. Find, with proof, the minimal possible number of cells in a CW-decomposition of the torus $S^1 \times S^1$.

   Solution: A CW-decomposition with 4 cells is standard: One 0-cell with two 1-cells $a, b$ and a 2-cell attached via the map $aba^{-1}b^{-1}$. However, there cannot be a decomposition with fewer cells, since the total homology has rank 4.

4. A *wedge sum* of two spaces $X \vee Y$ is obtained by taking the disjoint union $X \amalg Y$ and identifying one chosen point of $X$ with one chosen point of $Y$. Describe the universal cover of the wedge sum $\mathbb{R}P^2 \vee \mathbb{R}P^2$.

   Solution: A union of copies of the unit sphere $x^2 + y^2 + z^2 = 1$ shifted by $(0, 0, 2k)$, $k \in \mathbb{Z}$.

5. (a) Is every continuous map $S^2 \to S^1 \times S^1$ homotopic to a constant map?

   (b) Is every continuous map $S^1 \times S^1 \to S^2$ homotopic to a constant map?

   Prove your answers.

Solution: (a) Since $\pi_1(S^2) = 0$, by the lifting theorem, the map would have to lift to the universal cover $\mathbb{R}^2$ of $S^1 \times S^1$, which is contractible. Thus, the map is homotopic to a constant map.

(b) Consider, for example, the map given by contracting the two 1-cells in the standard CW-decomposition mentioned above. This map is not homotopic to a constant map, since it induces an isomorphism in $H_2$.