ALGEBRAIC TOPOLOGY QR
JANUARY 2021

All maps below are assumed to be continuous.

(1) Let Σ_2 be the compact oriented surface of genus 2 (without boundary). Take a disc $D \subset \Sigma_2$ centered at a point $p \in \Sigma_2$, let $S^1 \subset D$ be a circle that goes around the origin once. Let X be obtained from Σ_2 by collapsing this copy of S^1 to a point. Calculate $H_*(X)$.

(2) Let G be a topological space admitting a topological group structure, i.e., one has a continuous multiplication map $\mu : G \times G \to G$ and a continuous inversion map $\iota : G \to G$ that define a group structure on the set G. Assume that G is homeomorphic to a connected finite CW complex. Show that $\chi(G) = 0$ unless $G = \{1\}$.

(3) Consider the following properties of a connected finite CW complex X.

   (a) $\pi_1(X) \neq 0$ but $H_1(X) = 0$.

   (b) $H_1(X) = \mathbf{Q}$.

   For each of these properties, either construct an example satisfying the properties, or give a proof that none exists.

(4) Let $X = \mathbf{RP}^3$ and $Y = S^1 \vee S^1$.

   (a) Are all maps $f : X \to Y$ null-homotopic?

   (b) Are all maps $g : Y \to X$ null-homotopic?

   For each of the above, give a proof if the answer is "yes" and give an example if the answer is "no".

(5) Let $\pi : \mathbf{C}^3 - \{0\} \to \mathbf{CP}^2$ be the natural map, sending a point $x \in \mathbf{C}^3 - \{0\}$ to the line $\ell_x \in \mathbf{CP}^2$ connecting x to 0 in $\mathbf{C}^3$. Does π admit a section (i.e., a right-inverse)?

Solutions

(1) We give two proofs: one computational, one direct.

For the computational proof, we use the long exact sequence

$$...\tilde{H}_*(S^1) \to \tilde{H}_*(\Sigma_2) \to \tilde{H}_*(X) \to ....$$

of reduced homology groups for the good pair (Σ_2, S^1). This then gives:

- $H_0(X) = \mathbf{Z}$: X is non-empty and connected.
- $H_i(X) = 0$ for $i \geq 3$: this follows from the LES as $H_i(\Sigma_2) = 0$ for $i \geq 3$ and $H_i(S^1) = 0$ for $i \geq 2$.

To calculate H_1 and H_2, we write out the relevant piece of the long exact sequence:

$$0 = H_2(S^1) \to H_2(\Sigma_2) \to H_2(X) \to H_1(S^1) \xrightarrow{a} H_1(\Sigma_2) \to H_1(X) \to 0.$$

But we must have $a = 0$: the map $S^1 \to \Sigma_2$ is null-homotopic as it factors through a contractible space (the disc). So the above sequence breaks into two pieces:

$$0 \to H_2(\Sigma_2) \to H_2(X) \to H_1(S^1) \to 0 \quad \text{and} \quad H_1(\Sigma_2) \simeq H_1(X).$$

This gives the remaining calculations:

- $H_2(X) = \mathbf{Z}^2$ as both $H_1(S^1)$ and $H_2(\Sigma_2)$ are copies of $\mathbf{Z}$ (and short exact sequences of abelian groups with last term free split).
- $H_1(X) = \mathbf{Z}^4$ by the known calculation $H_1(\Sigma_2) = \mathbf{Z}^4$.

For the direct proof, one could simply observe that $X \simeq \Sigma_2 \vee S^2$. Indeed, this holds true with Σ_2 replaced by the small disc $D \subset \Sigma_2$ that contains the S^1 being collapsed, and thus follows as the assertion is local near the point. This then recovers the above calculations using the standard formula for the homology of a wedge. (We omit the details.)

(2) We use the Lefschetz fixed point formula. Assuming $G \neq \{1\}$, pick some $1 \neq g \in G$. Then left multiplication by g gives a continuous automorphism $f_g : G \to G$ which has no fixed points. The Lefschetz fixed point theorem then implies that the Lefschetz number of f_g vanishes, i.e.,

$$\Lambda_{f_g} := \sum_i (-1)^i \mathrm{Tr}(f_g | H_*(X)) = 0.$$

On the other hand, since G is path connected, we can pick a map $\phi : I \to G$ such that $\phi(0) = 1$ and $\phi(1) = g$. This gives a map $H : I \times G \to G$ via $H(t, h) = \mu(\phi(t), h)$ which can be regarded as a homotopy between $H(0, -) = \mathrm{id}$ and $H(1, -) = f_g$. Thus, $\Lambda_{f_g} = \Lambda_{\mathrm{id}}$. But the latter is $\chi(G)$, so we conclude that $\chi(G) = 0$.

(3) (a) Such an example exists. Let G be a finite non-abelian simple group (e.g., A_5). Choose a set of generators $g_1, ..., g_n \in G$, giving a presentation $G = \langle g_1, ..., g_n \mid r_1, ..., r_k \rangle$ of G (so the r_i's are words in the g_j's). We claim that there exists a finite connected CW complex X with $\pi_1(X) = G$. Granting this, we are done by Hurewicz's theorem: $H_1(X) = \pi_1(X)^{ab} = G^{ab} = 0$ as G is simple and non-abelian, but $\pi_1(X) \neq 0$. We build such a finite CW complex X as follows:

- X has a single 0-cell $\{x\}$.
- X has n distinct 1-cells, labelled by the generators $g_1, ..., g_n$ of G. As there is a single 0-cell, this simply means that the 1-skeleton X^1 is $\vee_{j=1}^n S^1$ (with j-th summand corresponding to g_j), whence $\pi_1(X^1)$ is the free group on n-generators $g_1, ..., g_n$.
- X has r distinct 2-cells $D_1, ..., D_k$ with the i-th attaching map $S^1 := \partial D_i \to X^1$ being any map whose homotopy class in $\pi_1(X^1)$ is given by the word r_i.

For the above CW complex, using SvK shows that $\pi_1(X) = \langle g_1, ..., g_n \mid r_1, ..., f_r k \rangle = G$, as wanted.

(b) This cannot happen: if X is a finite CW complex, then each $H_i(X)$ is a finitely generated abelian group (e.g., via CW homology).

(4) (a) Yes. Given such a map, the image of $\pi_1(X) = \mathbf{Z}/2$ in $\pi_1(Y) = \mathbf{Z} * \mathbf{Z}$ must be trivial as subgroups of free groups are free and hence cannot contain torsion elements. By covering space theory, this implies that f can lifted to the universal cover $\tilde{Y} \to Y$. But the universal cover $\tilde{Y}$ is contractible. Any map factoring through a contractible space is null-homotopic, so we win.

(b) No. As $\pi_1(X) = \mathbf{Z}/2 \neq 0$, there are certainly maps $S^1 \to X$ which induce a nonzero map on π_1 and are thus not null-homotopic. We can compose any such map with the natural codiagonal map $S^1 \vee S^1 \to S^1$ (i.e., identity on each S^1 on the source) to obtain a map $f : Y \to X$ which is also nonzero on π_1, and thus not null-homotopic.

(5) No. If a right inverse existed, then $H_*(\pi)$ would also have a right inverse, and hence be surjective. Now $\mathbf{C}^3 - \{0\} = \mathbf{R}^6 - \{0\}$ is homotopic to S^5 and hence has no H_2. On the other hand, the standard CW decomposition for $\mathbf{CP}^2$ has exactly 1 cell in dimension 0, 2 and 4 (and nothing else), so $H_2(\mathbf{CP}^2) = H_2(S^2) = \mathbf{Z} \neq 0$. Thus, $H_*(\pi)$ cannot be surjective, so π cannot admit a left-inverse.