ALGEBRAIC TOPOLOGY QR
MAY 2021

All maps below are assumed to be continuous.

(1) Let $i : \partial M \to M$ be the inclusion of the boundary of the Mobius strip M into M. Describe the induced map $\pi_1(i)$ on fundamental groups as a map of abelian groups.

(2) Let $f : X \to Y$ be a covering space of path-connected topological spaces. For each of the following constraints on f, X or Y, determine if such a covering space f exists. If one exists, construct it; if not, explain why.

   (a) $Y = S^1 \times S^1 \times S^1$ and f is not a regular covering space. (Recall that *regular* covering spaces are sometimes also called *Galois* covering spaces.)

   (b) $X = \mathbf{RP}^3$ and Y is homotopy-equivalent to a graph.

(3) Fix some $n \geq 1$. Assume we are given a continuous automorphism $f : \mathbf{CP}^n \to \mathbf{CP}^n$ of order 5. Show that f must have a fixed point.

(4) For any integer $g \geq 1$, let Σ_g be a compact oriented surface of genus g. Show that there are no covering spaces $f : \Sigma_4 \to \Sigma_3$.

(5) Let X be the space obtained by glueing two copies of S^3 together along a (smoothly embedded) closed submanifold diffeomorphic to the torus $T = S^1 \times S^1$, i.e., $X = S^3 \cup_T S^3$. Calculate $H_*(T)$.

Solutions

(1) Consider the unit square $S = [0,1]^2$. Write $\{e_1, e_2, e_3, e_4\}$ for the 4 oriented edges of S in standard order. (Draw picture.) Then M is obtained from S by identifying e_1 with e_3 in the opposite orientation; thus, the boundary ∂M is spanned by $e_2 \cup e_4$, and we have $S^1 = \partial M$, so $\pi_1(\partial M) = \mathbf{Z}$. On the other hand, projecting away the direction parallel to the glued edges gives a map $f : M \to S^1$ that is a homotopy-equivalence, so $\pi_1(M) = \mathbf{Z}$ via f. The composition $S^1 = \partial M \xrightarrow{i} M \xrightarrow{f} S^1$ is a degree 2 map: each of e_2 and e_4 goes around the target circle exactly once and in the same direction. Consequently, we learn that $\pi_1(i)$ is the map $\mathbf{Z} \xrightarrow{2} \mathbf{Z}$.

(2) (a) No, such a covering space does not exist. If it existed, then $f_*(\pi_1(X))$ would be non-normal subgroup of $\pi_1(Y)$. But $\pi_1(Y) = \mathbf{Z}^3$ is abelian, so all subgroups are normal.

(b) No, such a covering space does not exist. Indeed, the subgroup $\pi_1(X) \simeq f_*\pi_1(X)$ of $\pi_1(Y)$ would have to be a free group: subgroups of free groups are free, and Y is a homotopy-equivalent to a graph and hence has a free fundamental group. But $\pi_1(X) = \mathbf{Z}/2$ is not a free group.

(3) Recall that $H_*(\mathbf{CP}^n)$ is a copy of $\mathbf{Z}$ for $* \in \{0, 2, 4, ..., 2n\}$ and 0 in all other degrees. We claim that $H_*(f) = \mathrm{id}$. Granting this, it follows that $L(f) = \chi(\mathbf{CP}^n) = n+1 > 0$, so f must have a fixed point by the Lefschetz fixed point theorem. To prove $H_*(f) = \mathrm{id}$, we simply observe that $H_*(-)$ is a functor, so $H_*(f)$ is an order 5 automorphism of $\mathbf{Z}$ in all even degrees $\leq 2n$ (and obviously the identity in all other degrees as the homology vanishes in other degrees). But $\mathbf{Z}$ has no such automorphisms, so $H_*(f) = \mathrm{id}$, as wanted.

(4) Say we had a covering space $f : \Sigma_4 \to \Sigma_3$. Then f would have to have finite degree since Σ_4 is compact. If we call the degree d, then we have the Euler characteristic formula

$$d * \chi(\Sigma_3) = \chi(\Sigma_4).$$

But $\chi(\Sigma_g) = 2 - 2g$. Plugging this in above gives

$$d * (2 - 6) = 2 - 8$$

which simplifies to

$$-4d = -6,$$

which is impossible since d is an integer.

(5) The glueing description of X gives a Mayer-Vietoris LES that looks as follows:

$$...H_*(T) \xrightarrow{\alpha} H_*(S^3) \oplus H_*(S^3) \to H_*(X) \to H_{*-1}(T) \to ...$$

To use this, we must understand α. The map α_0 is the inclusion $\mathbf{Z} \xrightarrow{(1,-1)} \mathbf{Z} \times \mathbf{Z}$ by connectedness considerations, so $H_0(X) = \mathbf{Z}$. Moreover, $H_{>0}(S^3)$ is concentrated in degree 3, while $H_{>0}(T)$ is concentrated in degrees 1 and

2. Thus, $\alpha_i = 0$ for $i > 0$. Using the long exact sequence, this gives:

$$H_1(X) = 0 \quad \text{and} \quad H_2(X) = H_1(T) = \mathbf{Z} \times \mathbf{Z}.$$

Moreover, for $H_3(X)$, we obtain a short exact sequence

$$0 \to H_3(S^3)^2 \to H_3(X) \to H_2(T) \to 0.$$

Since $H_2(T) = H_3(S^3) = \mathbf{Z}$ and every short exact sequence of free abelian groups splits, we conclude that

$$H_3(X) = \mathbf{Z}^3.$$

All higher groups vanish as $H_{>3}(S^3) = H_{>2}(T) = 0$.