ALGEBRAIC TOPOLOGY QR
JANUARY 2021

All maps below are assumed to be continuous.

(1) Let Σ_2 be the compact oriented surface of genus 2 (without boundary). Take a disc $D \subset \Sigma_2$ centered at a point $p \in \Sigma_2$, let $S^1 \subset D$ be a circle that goes around the origin once. Let X be obtained from Σ_2 by collapsing this copy of S^1 to a point. Calculate $H_*(X)$.

(2) Let G be a topological space admitting a topological group structure, i.e., one has a continuous multiplication map $\mu : G \times G \to G$ and a continuous inversion map $\iota : G \to G$ that define a group structure on the set G. Assume that G is homeomorphic to a connected finite CW complex. Show that $\chi(G) = 0$ unless $G = \{1\}$.

(3) Consider the following properties of a connected finite CW complex X.

  (a) $\pi_1(X) \neq 0$ but $H_1(X) = 0$.

  (b) $H_1(X) = \mathbf{Q}$.

  For each of these properties, either construct an example satisfying the properties, or give a proof that none exists.

(4) Let $X = \mathbf{RP}^3$ and $Y = S^1 \vee S^1$.

  (a) Are all maps $f : X \to Y$ null-homotopic?

  (b) Are all maps $g : Y \to X$ null-homotopic?

  For each of the above, give a proof if the answer is "yes" and give an example if the answer is "no".

(5) Let $\pi : \mathbf{C}^3 - \{0\} \to \mathbf{CP}^2$ be the natural map, sending a point $x \in \mathbf{C}^3 - \{0\}$ to the line $\ell_x \in \mathbf{CP}^2$ connecting x to 0 in $\mathbf{C}^3$. Does π admit a section (i.e., a right-inverse)?