Differential Topology QR Exam
August 16, 2021

M denotes a C^∞ manifold of dimension n.
$\mathfrak{X}(M)$ is the space of all smooth vector fields on M.
All items will be graded independently of each other, to the extent possible.

Problem 1.- Assume M is connected. Prove that between any two points there exists a smooth curve connecting them.

Problem 2.- On vector fields:

(1) Let $X \in \mathfrak{X}(M)$, and assume that $\exists \epsilon > 0$ such that $\forall p \in M$ the integral curve of X through p is defined $\forall t \in (-\epsilon, \epsilon)$. Prove that X is complete.
(2) Use (1) to show that every vector field on a compact manifold is complete.

Problem 3.- Two unrelated questions on Lie groups:

(1) Let $U \subset G$ be a neighborhood of the identity where G is a Lie group. Show that there exists V a neighborhood of the identity such that $V \subset U$ and $\forall g, h \in V$ $gh^{-1} \in U$.
(2) Show that every Lie group G is orientable, and that not all orientable manifolds admit a Lie group structure.

Problem 4.- Let α be a smooth one-form on M. Assume that $\forall p \in M$ $\alpha_p \neq 0$.

(1) Show that $\mathcal{N} := \{(p, v) \in TM \mid \alpha_p(v) = 0\}$ is a submanifold of the tangent bundle TM.
(2) Assume that $d\alpha = 0$. Prove that $\forall p \in M$ there exists a regular submanifold $S \subset M$ such that $p \in S$ and $\forall q \in S$ $T_qS = \ker \alpha_q$.

Problem 5.- Let $f : M \times W \to \mathbb{R}$ be a smooth function, where $W \subset \mathbb{R}^k$ is open. For each $(p, w) \in M \times W$, define the partial differential $d_M f_{(p,w)} \in T_p^*M$ by

$$d_M f_{(p,w)}(\dot\gamma(0)) = \frac{d}{dt} f(\gamma(t), w)|_{t=0}$$

for each smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$.

Assume that zero is a regular value of the map $\Phi : M \times W \to \mathbb{R}^k$ defined as

$$\Phi(p, w) = \left(\frac{\partial f}{\partial w^1}(p, w), \ldots, \frac{\partial f}{\partial w^k}(p, w) \right).$$

(1) Let $\mathcal{C} := \Phi^{-1}(0)$. Explain why $\mathcal{C}$ is a submanifold and compute its dimension. If $(p, x) \in \mathcal{C}$ and $(x^1, \ldots, x^n)$ are coordinates in a neighborhood of p, write equations for the components $(\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^k)$ of vectors in $T_{(p,w)}\mathcal{C}$ in the coordinates $(x^1, \ldots, x^n, w^1, \ldots, w^k)$.
(2) Show that the map

$$F : \mathcal{C} \to T^*M, \qquad F(p, w) = \big(p, d_M f_{(p,w)}\big)$$

is an immersion.