THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

May, 2020: Algebraic Topology

1. Which of the following groups are fundamental groups of compact surfaces without boundary? For those which are, classify the surface:

   (a) $\langle a, b, c \mid abca^{-1}b^{-1}c \rangle$

   (b) $\langle a, b, c, d \mid abcda^{-1}b^{-1}c^{-1}d^{-1} \rangle$

   (c) $\langle a, b, c \mid abcb^{-1}a^{-1}c \rangle$.

2. Describe a set of free generators of the subgroup of the free group on two generators a, b generated by b and all the conjugates of a^2, b^2, and $(ab)^3$. Is this a normal subgroup?

3. Let S^1 be the unit circle in $\mathbb{C}$. Let a space X be obtained from $S^1 \times [0, 1]$ by identifying $(x, \epsilon) \sim (ix, \epsilon)$ for all $x \in S^1$, $\epsilon = 0, 1$, with the quotient topology. Compute $\pi_1(X)$ in terms of generators and defining relations.

4. Let T be the set of all 4-tuples $(x, y, z, t) \in \mathbb{C}^4 \smallsetminus \{(0, 0, 0, 0)\}$ satisfying

   $$xy + zt = 0$$

   and let X be the quotient of T by identifying $(x, y, z, t) \sim (\lambda x, \lambda y, \lambda z, \lambda t)$ for all $\lambda \in \mathbb{C} \smallsetminus \{0\}$. Let Y be the subset of X consisting of points represented by tuples of the form $(x, y, z, 0)$ and let Z be the one point subset represented by $(0, 0, 1, 0)$.

   (a) Prove that $\emptyset \subset Z \subset Y \subset X$ (possibly with some identity inclusions inserted) is a CW filtration of X.

   (b) Compute the homology of X.

5. Let Q be the quotient of $\mathbb{C}^n$ by identifying $(x_1, \dots, x_n) \sim (\lambda x_1, \dots, \lambda x_n)$ for all $\lambda \in \mathbb{C} \smallsetminus \{0\}$ with $|\lambda| = 1$. For what values of $n \in \mathbb{N}$ is Q a topological manifold without boundary?

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Qualifying Review Examination in Topology/Geometry

May 15, 2020: Morning Session, 9:00 to 12:00 noon.

Take "smooth" and "differentiable" here to mean $\mathcal{C}^\infty$.

1. The following questions are True/False. State whether the given statement is always true ("True") or only sometimes or never true ("False"), and give a brief reason why it is true, or a counter-example if your answer is "False".

   a) If M is a compact manifold with boundary $\partial M = N$, and N is orientable, then so is M.

   b) If $f : M^5 \to N^3$ is a smooth mapping of compact, differentiable manifolds of the indicated dimensions, and $f(M)$ contains an open set of N, then for some $y \in N$, $f^{-1}(y) \subset M$ is a differentiable manifold of dimension 2.

   [10]

2. Let f be a smooth function in a neighborhood of $0 \in \mathbb{R}^2$. Suppose $df(0) = 0$, *i.e.*, $\frac{\partial f}{\partial x_i}(0) = 0, i = 1, 2$. Show that the number of positive, negative and zero eigenvalues of the Hessian of f at 0, $Hess(f)(0) = (\frac{\partial^2 f}{\partial x_i \partial x_j})(0)$, are independent of the choice of local coordinates (x_1, x_2) at 0.

   [10]

3. Let $M = T^2 = S^1 \times S^1$, the 2-torus. Calculate the first DeRham cohomology group $H^1_{DR}(M)$ of M. (*Do not just quote the DeRham isomorphism to do this.*)

   [10]

4. A smooth two form β on a manifold M is called non-degenerate if the $2n$ form $\beta^n = \beta \wedge \cdots \wedge \beta$ (n factors) is nowhere 0, where $\dim M = 2\text{n}$.

a) Show that the form

$$\omega = \frac{dx \wedge dy}{(1+|z|^2)^2},$$

on $\mathbb{R}^2 = \mathbb{C}$, where $z = x + i\,y$, has a smooth extension Ω to S^2 which is non-degenerate.

b) If $f(z) := \bar{z}^3 + 2\bar{z}$, what is $\int_{S^2} f^*\Omega$?

c) Show that there is no *closed* non-degenerate two form on S^4.
(*You may use that* $H^2_{DR}(S^{2n}) = 0$, *for* $n > 1$.)

[10]

5. Consider $\mathbb{R}^4 = \mathbb{C}^2$ with coordinates $z_j = x_j + i\,y_j, j = 1, 2$, and consider the vectorfields

$$\begin{aligned}
\xi_1 &= y_1 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial y_2}, \\
\xi_2 &= -x_2 \frac{\partial}{\partial x_1} - y_2 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2}, \\
\xi_3 &= -y_2 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y_2}.
\end{aligned}$$

a) Show that each $\xi_j, j = 1, 2, 3$, is tangent to the unit sphere

$$S^3 = \{(x_1, y_1, x_2, y_2) \,|\, x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1\} \subset \mathbb{R}^4.$$

(*Please just show that* ξ_1 *is tangent to* S^3*; the other cases are similar.*)

b) Show that $[\xi_1, \xi_2] = -2\,\xi_3, [\xi_2, \xi_3] = -2\,\xi_1, [\xi_3, \xi_1] = -2\,\xi_2$.
(*Please just show that* $[\xi_1, \xi_2] = -2\,\xi_3$*; the other cases are similar.*)

c) Show that $\mathfrak{g} = \mathbb{R}\xi_1 + \mathbb{R}\xi_2 + \mathbb{R}\xi_3$ is a Lie algebra.

d) What is a Lie group whose Lie algebra is $\mathfrak{g}$?
(*No proof necessary.*)

[10]