THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

January 6, 2020: Algebraic Topology

1. Let D^2 be the unit disk in $\mathbb{C}$, and let T_n be the manifold with $n+1$ boundary components obtained from D^2 by removing all complex numbers of distance $< \lambda$ from any of the points
$$\frac{1}{2}e^{2\pi ik/n},\ k \in \mathbb{Z}$$
where $\lambda < min(|e^{2\pi i/n} - 1|/2, 1/2)$. Now suppose that T_n is embedded into a compact closed oriented smooth surface X without boundary of genus m as a smooth submanifold, and a manifold Y is obtained from X by removing the interior of T_n, and attaching a disk to each of the $n+1$ resulting boundary components. Assume further that the manifold Y is also connected.
   (a) For which values of $m, n \in \mathbb{N}_0$ is this possible?
   (b) What can we say about the genus of Y?

2. Describe a set of free generators of the subgroup of the subgroup of the free group on two generators a, b generated by all conjugates of $aba^{-1}b^{-1}$.

3. Consider the manifold T_n from Problem 1. Let its boundary components be $C_0, \dots, C_n$. Let $S^1 \subset \mathbb{C}$ be the unit sphere. Let $k \in \mathbb{Z}$. Consider the space X obtained from
$$T_n \amalg (D^2 \times \{0, \dots, n\})$$
by identifying, for $x \in D^2$, (x, i) with $\phi_i(x)$ where $\phi_i : S^1 \to C_i$ are maps of degree k. Compute $\pi_1(X)$ in terms of generators and defining relations.

4. For $n > 1$, let X be the pushout of the diagram
$$\begin{array}{ccc} \mathbb{R}P^{n-1} & \xrightarrow{\subset} & \mathbb{R}P^n \\ {\scriptstyle\subset}\downarrow & & \\ \mathbb{R}P^n. & & \end{array}$$
Compute the homology of X.

5. The *unreduced suspension* $\widetilde{X}$ of a space X is obtained from $X \times [0, 1]$ by identifying $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all choices of points $x, y \in X$. If S^n is the n-sphere, $n > 0$, compute the homology of the unreduced suspension of $S^n \times \{0, \dots, k\}$.

THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

January 6, 2020: Afternoon Session, 2:00 P.M. to 5:00 PM

1. Let $\gamma_{2,p}, p$ an odd prime, be the map from $S^3 := \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2 + |w|^2 = 1\}$ to itself given by $(z,w) \to \gamma_{2,p} \cdot (z,w) = (-z, e^{\frac{2\pi i}{p}} w)$, and let $\Gamma := \langle \gamma_{2,p} \rangle$, the cyclic group generated by $\gamma_{2,p}$. Let $L(2,p) = S^3/\Gamma$ be the quotient space. Show that one may make $L(2,p)$ a smooth manifold so that the quotient map $(z,w) \to (z,w) \bmod \Gamma$ is differentiable.

2. Let $S \subset M_2(\mathbb{R})$ be the set of singular 2×2 matrices, i.e.,

$$S = \{A \in M_2(\mathbb{R}) \,|\, \det A = 0\}.$$

Show that S is a smooth submanifold in $M_2(\mathbb{R})$ away from the origin, the zero matrix in $M_2(\mathbb{R})$. Let $tr : S \to \mathbb{R}$ be the trace map restricted to S. Show that every fiber $tr^{-1}(t)$ is a smooth manifold of S, for all $t \neq 0$. Show that each $t \neq 0$ is a regular value of tr restricted to S.

3. Let M be a compact manifold, and let $E \to M$ be a real vector bundle of rank n on M. Show there is a trivial vector bundle $M \times \mathbb{R}^N$ and a surjective bundle map $\rho : M \times \mathbb{R}^N \to E$.

4. Let

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

a non-singular skew-symmetric matrix in $GL(2n,\mathbb{R})$. Let ω be the non-degenerate skew-symmetric 2-form on $\mathbb{R}^{2n}$ given by $\omega(v,w) = v^T J w$, for $v, w \in \mathbb{R}^{2n}$, and let $Sp(2n,\mathbb{R})$ be the subgroup of $GL(2n,\mathbb{R})$ which preserves ω, i.e., the set

$$\{A \in GL(2n,\mathbb{R}) \,|\, \omega(Av, Aw) = \omega(v,w), \forall v, w \in \mathbb{R}^{2n}\}.$$

Show that $Sp(2n,\mathbb{R})$ is a Lie group. What is the dimension of $Sp(2n,\mathbb{R})$? What is the Lie algebra of $Sp(2n,\mathbb{R})$?

5. a. Show that the smooth 2-form $\omega = \frac{dx \wedge dy}{(1+|z|^2)^2}$ on $\mathbb{C}$, where $z = x + iy$, as usual, has a smooth extension to the Riemann sphere $\mathbb{CP}^1 \cong S^2$.

b. Let $f(z) = 3\bar{z}^3 - 2\bar{z} + 5$, a map from $\mathbb{C} \to \mathbb{C}$, where $\bar{z} = x - iy$. Show that f extends smoothly as a map from $\mathbb{CP}^1 \to \mathbb{CP}^1$. Calculate

$$\int_{\mathbb{CP}^1} f^* \omega.$$