ALGEBRAIC TOPOLOGY QR
AUGUST 2020

All maps below are assumed to be continuous.

(1) Let M be the Möbius band. Consider the pushout X of

$$\begin{array}{ccc} S^1 = \partial M & \longrightarrow & M \\ \big\downarrow & & \\ S^1 & & \end{array}$$

where the horizontal map is the inclusion of the boundary, and the vertical map is a degree 2 covering space. Describe each $H_i(X)$ as an abelian group.

(2) Consider the following property of spaces X equipped with a base point $x \in X$:

($*$) For any covering space $Y \to X$, if Y is connected, then so is the preimage $f^{-1}(X - \{x\})$.

For each of the following spaces X, determine if they satisfy ($*$) with respect to any base point. (If yes, then give a proof; if not, then give an example.)

(a) $X = S^1$.

(b) $X = \Sigma_2$ (the compact closed oriented surface of genus 2).

(3) Let $f : S^4 \to S^4$ be a map with the property that $f(x) = f(y)$ if y is the antipode of x. Show that $H_4(f) = 0$.

(4) Show that a finite group G of order 7 cannot act freely on $\mathbf{CP}^5$.

(5) For each of the following cases, determine if there exists a covering space $f : X \to Y$. (If yes, then construct it; if not, then give a proof.)

(a) X is homotopy equivalent to $S^1 \times S^1$ and Y homotopy equivalent to $S^1 \vee S^1$.

(b) X is homotopy equivalent to $S^1 \vee S^1$ and Y homotopy equivalent to $S^1 \times S^1$.