MAY 2019

SOLUTIONS

1. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is both connected and simply connected.

Solution: To see that $\mathbb{R}^n \setminus M$ is path-connected, let $p, q \in \mathbb{R}^n \setminus M$ and let $c(t)$ be a path in $\mathbb{R}^n$ with $c(0) = p$ and $c(1) = q$. By the parametric transversality theorem, we can perturb the interior of c so that c intersects M transversally. However since $\dim M + \dim c < n$, intersecting transversally means having empty intersection. So we have found a path from p to q which does not touch M, showing $\mathbb{R}^n \setminus M$ is path-connected.

To see that $\mathbb{R}^n \setminus M$ is simply connected, let $c_1(t)$ and $c_2(t)$ be two closed loops in $\mathbb{R}^n \setminus M$. Since $\mathbb{R}^n$ is simply connected, there is a homotopy $F(s,t) = c_s(t)$ between c_1 and c_2. By the parametric transversality theorem, we can perturb the surface $F(s,t)$ so that it intersects M transversally. As before, the dimension of M is small enough such that transverse intersection means empty intersection.

2. M is a smooth manifold of dimension n, and ω is a smooth k-form on M where $k \geq 1$.

(1) If k is odd, show $\omega \wedge \omega = 0$.

Solution: If α is a k form and β is an l form then $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$. Consequently, $\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega = -\omega \wedge \omega$ since k odd implies k^2 odd. Thus $\omega \wedge \omega = 0$.

(2) What are the minimal values of k and n so that $\omega \wedge \omega$ is possibly nonzero. Give such an example.

Solution: By the previous question, $k = 2$ is the smallest potential possible value for k. If so, $\omega \wedge \omega$ is a 4-form, so $n \geq 4$ or else $\alpha \wedge \alpha$ would be 0. The values $k = 2$ and $n = 4$ can be realized by the standard symplectic form on $\mathbb{R}^2 \times \mathbb{R}^2$, namely $\alpha = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. Indeed, $\alpha \wedge \alpha$ is some multiple of $dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$, which is a nowhere vanishing volume form.

(3) Let α be a closed differential 2-form on S^4. Prove that $\alpha \wedge \alpha$ vanishes at some point.

Solution: Since $H^2_{DR}(S^4) = 0$, we know α is exact, i.e. $\alpha = d\beta$ for some 1-form β. Note

$$d(\beta \wedge \alpha) = \beta \wedge d\alpha + d\beta \wedge \alpha = \alpha \wedge \alpha.$$

By Stokes' theorem,

$$\int_{S^4} \alpha \wedge \alpha = \int_{S^4} d(\beta \wedge \alpha) = 0.$$

Since $\alpha \wedge \alpha$ is a 4-form on S^4, we know $\alpha \wedge \alpha = f\omega$, where f is a smooth function and ω is a nowhere vanishing volume form on S^4. If f is never 0, then by continuity, f cannot change sign, and so $\int_{S^4} f\omega \neq 0$, contradicting the above computation.

3. Show that the real cubic surface given by

$$S = \{[x : y : z : w] \in \mathbb{R}P^3 \ : \ x^3 + y^3 + z^3 + w^3 = 0\}$$

is a smooth submanifold of $\mathbb{R}P^3$, and compute its dimension.

Solution: Suppose $[x : y : z : w]$ is such that $w \neq 0$. Then in local coordinates, the defining equation for S becomes $f(x, y, z) = x^3 + y^3 + z^3 - 1 = 0$. Observe $x = y = z = 0$ does not satisfy the equation. Hence $df = (3x, 3y, 3z)$ cannot be zero on S. By the regular value theorem, we have shown $S \cap \{w \neq 0\}$ is a smooth submanifold of $\mathbb{R}P^3$ of codimension 1, or dimension 2. An analogous argument works for the three remaining charts $\{x \neq 0\}$, $\{y \neq 0\}$, $\{z \neq 0\}$.

4.

(1) Show that the subset M of $\mathbb{R}^3$ defined by the equation

$$(1 - z^2)(x^2 + y^2) = 1$$

is a smooth submanifold of $\mathbb{R}^3$.

Solution: Let $f(x, y, z) = (1 - z^2)(x^2 + y^2)$. We want to show 1 is a regular value of f. If $(x, y, z) \in f^{-1}(1)$, we must have $1 - z^2 \neq 0$. We have

$$df = (2x(1 - z^2), 2y(1 - z^2), -2z(x^2 + y^2)).$$

If this is the 0 vector, then $1 - z^2 \neq 0$ implies $x = y = 0$. But $f(0, 0, z) = 0 \neq 1$, contradiction. So df has full rank for $(x, y, z) \in f^{-1}(1)$, which means $f^{-1}(1)$ is a smooth submanifold of $\mathbb{R}^3$.

(2) Define a vector field on $\mathbb{R}^3$ by

$$V = z^2 x \frac{\partial}{\partial x} + z^2 y \frac{\partial}{\partial y} + z(1 - z^2) \frac{\partial}{\partial z}.$$

Show that the restriction of V to M is a tangent vector field to M.

Solution: The vector field V is tangent to M iff the dot product $\mathrm{grad} f \cdot V$ is 0. We check

$$\mathrm{grad} f \cdot V = (2x(1 - z^2), 2y(1 - z^2), -2z(x^2 + y^2)) \cdot (z^2 x, z^2 y, z(1 - z^2)) = 0,$$

using that $f(x, y, z) = 1$ on M.

(3) The family of maps $\phi_t(x, y, z) = (cx - sy, sx + cy, z)$ with $c = \cos(t)$ and $s = \sin(t)$ obviously restricts to a one-parameter family of diffeomorphisms of M. For each t, determine the vector field $(\phi_t)_* V$ on M.

Solution: (The family of maps ϕ_t restricts to M means if $f(x,y,z) = 1$, then $f(\phi_t(x,y,z)) = 1$.)

Now $(\phi_t)_* V$ is simply the derivative matrix $D(\phi_t)$ applied to the vector V. We compute

$$D(\phi_t)(V) = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z^2 x \\ z^2 y \\ z(1-z^2) \end{pmatrix} = \begin{pmatrix} cz^2 x - sz^2 y \\ sz^2 z + cz^2 y \\ z(1-z^2) \end{pmatrix}.$$

5. Prove or disprove:

(1) Let M and N be two smooth manifolds. If the tangent bundles TM and TN are diffeomorphic, then M and N are diffeomorphic.

Solution: Let M be the cylinder $S^1 \times \mathbb{R}$ and let N be the Möbius band. Then M and N are not diffeomorphic since M is orientable but N is not.

We claim both TM and TN are diffeomorphic to $S^1 \times \mathbb{R}^3$. To see this, note $TM = T(S^1 \times \mathbb{R}) \cong TS^1 \times T\mathbb{R} \cong S^1 \times \mathbb{R} \times \mathbb{R}^2$. For TN, first note we can write

$$N = \{(x,y) \,|\, x \in [0, 2\pi],\ y \in \mathbb{R}\}/(0,y) \sim (2\pi, -y).$$

Then we can write

$$TN = \left\{ \left(x, y, a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right) \mid x \in [0,2\pi], y, a, b \in \mathbb{R} \right\} /(0,y,a,b) \sim (2\pi, -y, a, -b).$$

Let $f : TN \to S^1 \times \mathbb{R} \times \mathbb{C}$ be given by $f(x,y,a,b) = (x, a, e^{ix/2}(y+ib))$. Note $f(0,y,a,b) = f(2\pi, -y, a, -b)$ so f is a well-defined. Moreover, f is a diffeomorphism, which proves the claim.

(2) The tangent bundle of a 2-dimensional sphere S^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.

Solution: If $F : S^2 \times \mathbb{R}^2 \to TS^2$ is a diffeomorphism, then $F(p, x_0)$ for some $x_0 \neq 0$ in $\mathbb{R}^2$ defines a nowhere vanishing vector field on $\mathbb{R}^2$, contradicting the hairy ball theorem.