THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

August 31, 2019: Algebraic Topology.

1. Denote for a space X by CX the quotient $X \times [0, \infty)/(X \times \{0\})$. For which (connected) compact surfaces X is CX a topological manifold without boundary?

2. Let X be a space obtained from three copies of the Möbius strip by attaching their boundaries homeomorphically. Calculate $\pi_1(X)$ in terms of generators and defining relations.

3. Let $f : X \to Y$, $g : Z \to Y$ be connected coverings, where Y is a path-connected locally path-connected space. Let $X \times_Y Z = \{(x, z) \mid f(x) = g(z)\}$ with the subspace topology of the product topology. Let $p : X \times_Y Z \to Y$ be given by $(x, z) \mapsto f(x) = g(z)$.

   (a) Is p necessarily a covering?

   (b) Is $X \times_Y Z$ necessarily connected?

   (Prove your answers.)

4. Let a CW complex X be obtained from a k-sphere, $k \geq 1$, by attaching two $(k+1)$-cells along attaching maps of degrees $m, n \in \mathbb{Z}$. Calculate the homology of X.

5. For which $k \geq 1$ does there exist a continuous map $\mathbb{R}P^k \to S^k$ which is not homotopic to a constant map?

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September 3, 2016: Afternoon Session, 2:00 to 5:00.

1. Let $M(n,\mathbb{R})$ be the space of all real $n \times n$ matrices, and $Gl(n,\mathbb{R}) \subset M(n,\mathbb{R})$ be the subset of invertible matrices. Let $X \in Gl(n,\mathbb{R})$ and $B \in M(n,\mathbb{R})$. Show that

$$\frac{d}{dt}\det(X \cdot e^{tB})|_{t=0} = \det X \ \text{Trace}(B).$$

Show that $Sl(n,\mathbb{R}) := \{A \in M(n,\mathbb{R}) | \det A = 1\}$ is a closed submanifold of $Gl(n,\mathbb{R})$, of dimension $n^2 - 1$.

2. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and let C be the cubic surface defined by

$$C = \{(x,y,z) \in \mathbb{R}^3 \,|\, y^2 x = x^3 - xz^2\}$$

Define $X = S^2 \cap C$. Is X a smooth submanifold of $\mathbb{R}^3$?

3. Consider $\mathbb{R}^{2n}$ with coordinates $(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ and define the 1-form α by

$$\alpha = \sum_i y_i \, dx_i,$$

and the 2-form ω by

$$\omega = d\alpha.$$

Let V_x be the subspace $\{y = 0\} \subset \mathbb{R}^{2n}$ and $\iota_x : V_x \to \mathbb{R}^{2n}$ the inclusion, and similarly for V_y, ι_y. Show that the pull-backs $\iota_x^* \omega$ on V_x and $\iota_y^* \omega$ on V_y are identically zero. Let $S^* = \{(x,y) \,|\, y_1^2 + \cdots + y_n^2 = 1\}$. Show that the $2n-1$-form

$$\alpha \wedge (\omega)^{n-1} = \alpha \wedge \omega \wedge \cdots \wedge \omega \ \ (n-1 \text{ times})$$

is nowhere zero on the submanifold S^*. Write down a vector field ξ tangent to S^* which is not identically 0 such that for every vector field η tangent to S^*, we have $\omega(\xi, \eta) \equiv 0$.

4. Let M be a smooth manifold, $A \subset M$ a closed subset and $U \supset A$ an open neighborhood of A in M. Suppose that f is a smooth, real-valued function defined on U. Show that there is a smooth function $\tilde{f} : M \to \mathbb{R}$ such that $\tilde{f} \equiv f$ on a neighborhood of A.

5. Let $O(3) \subset Gl(3, \mathbb{R})$ be the 3×3 orthogonal group. Let $\omega = g^{-1} \cdot dg$ be the 3×3 matrix of one-forms on $O(3)$, where

$$g = \begin{pmatrix} g_{1,1} & \cdots & g_{1,3} \\ & & \\ g_{3,1} & \cdots & g_{3,3} \end{pmatrix}, \text{ and } dg = \begin{pmatrix} dg_{1,1} & \cdots & dg_{1,3} \\ & & \\ dg_{3,1} & \cdots & dg_{3,3} \end{pmatrix},$$

and the $g_{i,j}$ are the coordinate functions in $M(3, \mathbb{R})$. Finally, for $a \in O(3)$ fixed, let $L_a : O(3) \to O(3)$ be given by left multiplication, i.e.,

$$L_a(g) = a \cdot g.$$

Show that $L_a^* \omega = \omega$, *i.e.*, ω is left invariant.