1. Let $M^m \subset \mathbb{R}^n$ be a smooth submanifold of dimension $m < n-2$. Show that its complement $\mathbb{R}^n \setminus M$ is connected and simply connected.

2. M is a smooth manifold of dimension n, and ω is a smooth k -form on M where $k \geq 1$.
(a) If k is odd, show $\omega \wedge \omega = 0$.
(b) What are the minimal values of k and n such that $\omega \wedge \omega$ is possibly nonzero. Give such example.
(c) Let α be a closed differential two-form on S^4. Prove that $\alpha \wedge \alpha$ vanishes at some point.

3. Show that the subset defined by $S = \{[x : y : z : w] \in \mathbb{R}P^3 : x^3+y^3+z^3+w^3 = 0\}$ is an embedded submanifold of $\mathbb{R}P^3$, and compute its (real) dimension.

4. (a) Show that the subset M of $\mathbb{R}^3$ defined by the equation

$$(1-z^2)(x^2+y^2) = 1$$

is a smooth submanifold of $\mathbb{R}^3$.

(b) Define a vector field on $\mathbb{R}^3$ by

$$V = z^2 x \frac{\partial}{\partial x} + z^2 y \frac{\partial}{\partial y} + z(1-z^2)\frac{\partial}{\partial z}.$$

Show that the restriction of V to M is a tangent vector field to M.

(c) The family of maps $\phi_t(x,y,z) = (cx - sy, sx + cy, z)$ with $c = \cos t$ and $s = \sin t$ obviously restricts to a one-parameter family of diffeomorphisms of M. For each t, determine the vector field $(\phi_t)_* V$ on M.

5. Prove the following statement if it's true, or disprove if false:
(a) Let M and N be two smooth manifolds, if the tangent bundles TM and TN are diffeomorphic, then M and N are diffeomorphic.
(b) The tangent bundle of a 2-d sphere TS^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.

THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

May 10, 2019: Algebraic Topology

1. Let S^3 be the unit sphere in $\mathbb{R}^4$, and let ϕ_i, $i = 0, \dots, 4$ be the involution which reverses the signs of the first i coordinates. For which values of i is the quotient space of ϕ_i a topological manifold without boundary?

2. For which values of $g \geq 0$ is it true that for every number $h \geq g$ (g, h integers), a compact oriented surface X of genus g (without bounary) has a covering $f : Y \to X$ where Y is a compact oriented surface of genus h?

3. Let S^1 be the unit sphere in $\mathbb{C}$, let $T = S^1 \times S^1$ and let $T' = T/(S^1 \times \{1\})$. Let X be the "connected sum" of T and T', i.e. a space obtained by cutting out interiors of closed 2-disks from T and T', respectively, (disjoint from the singular point in case of T') and attaching the resulting spaces by the boundaries of the disks. Compute the fundamental group and homology of X.

4. Let F be a free group on two generators a, b and let $h : F \to \mathbb{Z}/2 \times \mathbb{Z}/2$ be an onto homomorphism. Is $Ker(h)$ a free group? If so, find its free generators.

5. Let X be a 2-dimensional CW-complex with one 0-cell, four 1-cells a, b, c, d and four 2-cells, attached along the loops a^2bc, ab^2d, ac^2d, bcd^2. Compute the homology of X.