THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

Algebraic topology special exam.

1. Let Z be a convex 10-gon in the plane with vertices $A_0, A_1, A_2, A_3, A_4, B_4, B_3, B_2, B_1, B_0$ appearing in this order on the boundary (oriented counter-clockwise). Let X be the topological space obtained from Z by gluing the line segments A_0A_1 with B_2B_3, B_0B_1 with A_2A_3, A_1A_2 with B_1B_2, A_3A_4 with B_3B_4, A_0B_0 with B_4A_4. All pairs of line segments are attached by linear maps with the vertices corresponding in the order listed (first to first, last to last).

   (a) Calculate $\pi_1(X)$.

   (b) Classify the surface X.

2. Prove that every CW-structure on $\mathbb{R}P^n$ has at least one cell in each dimension $0, 1, \dots, n$.

3. Let X be a graph with one vertex and two edges. Does there exist a connected covering $f : Y \to X$ which is regular and a connected covering $g : Z \to Y$ which is regular such that $fg : Z \to X$ is not a regular covering? Prove your answer.

4. Let $Z = (\mathbb{C} \smallsetminus \{e^{2k\pi i/5} \mid k \in \mathbb{Z}\}) \times [0,1]$. Let a space Y be obtained from Z by identifying $(z, 0)$ with $(ze^{2\pi i/5}, 1)$ for every $z \in \mathbb{C} \smallsetminus \{e^{2k\pi i/5} \mid k \in \mathbb{Z}\}$. Compute $\pi_1(Y)$.

5. Let
$$S^2 = \{\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$
$$D^3 = \{\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\},$$
   Let $f : S^2 \to S^2$ be a (continuous) map of degree k, and let $\pi : S^2 \to \mathbb{R}P^2$ be a covering. Let X be the pushout of the diagram
$$\begin{array}{ccc} S^2 & \xrightarrow{\pi\circ f} & \mathbb{R}P^2 \\ {\scriptstyle\subset}\big\downarrow & & \\ D^3. & & \end{array}$$
   Calculate the homology of X.

1. Let M be a smooth manifold and $C \subset O \subset M$, where C is a closed smooth submanifold and O is an open subset. Show that if $f : C \to \mathbb{R}$ is a smooth function, then there is a smooth function $\hat{f} : M \to \mathbb{R}$, such that $\hat{f}|_C = f$ and $supp(\hat{f}) \subset O$.

2. Let M be a smooth orientable manifold and let $\Psi : M \to \mathbb{R}$ be a smooth map. Show that if 0 is a regular value of Ψ, then $\Psi^{-1}(0) \subset M$ is also a smooth orientable manifold.

3. a)Give an example (with proof) of a homeomorphism $\mathbb{R} \to \mathbb{R}$ which is not a diffeomorphism.
b)Construct a smooth structure R' on $\mathbb{R}$ such that the identity function on $\mathbb{R}$ is not a diffeomorphism. Namely, let $(\mathbb{R}, R)$ be the standard smooth atlas, find $(\mathbb{R}, R')$ another atlas, $(\mathbb{R}, R) \xrightarrow{\psi} (\mathbb{R}, R')$, such that $\psi_{\mathbb{R}} = id$, but ψ is not a smooth map.

4. Condider the form $\omega = (x^2 + 2x + z)dy \wedge dz$ on $\mathbb{R}^3$. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere and $i : S^2 \to \mathbb{R}^3$ be the inclusion map.
a)Evaluate the integral $\int_{S^2} \omega$.
b)Construct a closed form θ on $\mathbb{R}^3$ s.t. $i^*\theta = i^*\omega$, or prove that such a form θ does not exist.

5. Denote $\mathcal{M}_{m\times n}(\mathbb{R})$ the space of $m \times n$ matrices with real-valued entries. Show that the subset $\mathcal{S}_k \subset \mathcal{M}_{m\times n}(\mathbb{R})$ of rank k matrices forms a dimension $k(m + n - k)$ smooth submanifold of $\mathcal{M}_{m\times n}(\mathbb{R})$. Here $1 \leq k < m \leq n$ and $k, m, n \in \mathbb{Z}$.