SEPTEMBER 2017

SOLUTIONS

1. Show that
$$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$
defines a nonzero deRham cohomology class of $\mathbb{R}^2 \setminus \{0,0\}$.

Solution: We need to show ω is closed but not exact.

Computing the exterior derivative gives
$$\begin{aligned} d\omega &= \left(\frac{\partial}{\partial x}\frac{x}{x^2+y^2} - \frac{\partial}{\partial y}\frac{-y}{x^2+y^2}\right) dx \wedge dy \\ &= \left(\frac{(x^2+y^2)-2x^2+(x^2+y^2)-2y^2}{(x^2+y^2)^2}\right) dx \wedge dy \\ &= 0, \end{aligned}$$
so ω is closed.

To show ω is not exact, we integrate ω around the unit circle C, which we will parametrize by $f(t) = (\cos(t), \sin(t))$ from $t = 0$ to 2π. It suffices to show the integral $\int_C \omega \neq 0$. Indeed, if ω were exact, then $\omega = dg$ for some smooth function g and then Stokes' theorem would give $\int_C dg = \int_{\partial C} g = 0$ since C has no boundary.

We have
$$\begin{aligned} f^*dx &= -\sin(t)dt \\ f^*dy &= \cos(t)dt. \end{aligned}$$
Writing x, y, dx, dy in terms of t, we get
$$\int_C \omega = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\,dt = 2\pi \neq 0.$$
Therefore, ω is not exact.

2. Any non-constant smooth function of a compact connected manifold has at least two critical points.

Solution: Let M be a compact connected manifold and let $f : M \to \mathbb{R}$ be a smooth function. Since M is compact and f is continuous, the function f must achieve both a maximum and minimum value on M. Suppose p is such that $f(p)$ is the maximum value of f. We claim f has a critical point at p. To see this, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a curve with $\gamma(0) = p$ (this is assuming M has no boundary, which is necessary or else $f(x) = x$ on $[0, 1]$ is a counter-example to the statement of the problem). Suppose for contradiction $\gamma'(t) \neq 0$. If $\gamma'(t) > 0$ then there is small enough $t > 0$ so that $f(\gamma(t)) > f(p)$ and if $\gamma'(0) < 0$, there is $t > 0$ so that $f(\gamma(-t)) > f(p)$, contradicting the maximality of $f(p)$. A similar argument shows that if $q \in M$ is such that $f(q)$ is the minimum value at f, then f must have a critical point at q.

3. For each $n \geq 1$, there is a diffeomorphism $(TS^n) \times \mathbb{R} \cong S^n \times \mathbb{R}^{n+1}$.

Solution: View S^n as $\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_1^2 + \cdots + x_n^2 = 1\}$. Let N_p be the normal vector field to S^n, i.e. associate to each $p = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^n$ the vector $(x_1, \ldots, x_{n+1})$. Define

$$\begin{aligned} F : (TS^n) \times \mathbb{R} &\to S^n \times \mathbb{R}^{n+1} \\ (p, v, \alpha) &\mapsto (p, v + \alpha N_p). \end{aligned}$$

This map is clearly smooth. It is bijective because any $w \in \mathbb{R}^{n+1}$ can be decomposed uniquely as $w = w_1 + w_2$ where w_1 is in the direction of N_p and w_2 is in the direction normal to N_p, i.e. in the direction of T_pS^n.

4. Assuming that every n-dimensional compact manifold M^n can be embedded into some $\mathbb{R}^N$, prove that we can choose $N = 2n + 1$. (Hint: Given a nonzero vector $v \neq 0$ in $\mathbb{R}^N$, once can define a parallel projection ϕ_v from $\mathbb{R}^N$ to the orthogonal complement of v. If $N > 2n + 1$, we can choose some v so that $\phi_v|_{M^n}$ is an embedding.)

Solution: We need to choose v so that ϕ_v is an injective immersion.

Injective: In order for ϕ_v to be injective, we need the line tv to pass through at most one point on M. Define

$$\begin{aligned} F : M \times M \setminus \Delta &\to \mathbb{R}P^{N-1} \\ (p, q) &\mapsto [p - q]. \end{aligned}$$

If $F(p, q) = [v]$, then the line tv will pass through both p and q, which means ϕ_v is not injective. If $[v]$ is not in the image of F, then the fiber of ϕ_v contains at most one point, i.e. ϕ is injective. We claim the image of F has measure 0 so long as $2n = \dim(M \times M \setminus \Delta) > \dim(\mathbb{R}P^{N-1}) = N - 1$. Indeed, if the dimensions satisfy this inequality, F cannot have any regular values, so every value of F must be a critical value. The set of critical values has measure 0 by Sard's theorem.

Immersion: In order for ϕ_v to be an immersion, we must have $d\phi_v$ is injective on tangent spaces. Since ϕ_v is linear, its derivative is itself. For the projection ϕ_v to be injective on tangent spaces, we must have that v is not parallel to any vector tangent to M. Define

$$\begin{aligned} G : TM \setminus \{(p, 0) | p \in M\} &\to \mathbb{R}P^{N-1} \\ (p, w) &\mapsto [w]. \end{aligned}$$

If v is not in the image of G, then v is not parallel to any vector tangent to M. Again, by Sard's theorem, the image of G has measure 0 so long as $2n = \dim(TM) > \dim(\mathbb{R}P^{N-1}) = N - 1$.

In conclusion, if $2n > N - 1$, we can choose v so that $[v] \in \mathbb{R}P^{N-1}$ is not in the image of F or G, guaranteeing the projection ϕ_v is an injective immersion, and hence an embedding.

5.

(1) Show that the space of orthogonal matrices

$$O(n) = \{A \in M_{n\times n}(R) \,|\, AA^T = Id\}$$

is a smooth submanifold of $M_{n\times n}(\mathbb{R})$.

Solution: Let $S(n)$ denote the space of $n \times n$ real symmetric matrices. Consider the function

$$\begin{aligned} F : M_{n\times n}(\mathbb{R}) &\to S(n) \\ A &\mapsto AA^T. \end{aligned}$$

By the regular value theorem, it suffices to show Id is a regular value of F. To this end, we compute the derivative

$$DF_A : T_A M_{n\times n}(\mathbb{R}) \to T_{Id} S(n)$$

for any $A \in O(n)$ and show it is surjective. Note we can identify $T_A M_{n\times n}(\mathbb{R}) \cong M_{n\times n}(\mathbb{R})$ and $T_F(A)S(n) \cong S(n)$. Let $B \in T_A M_{n\times n}(\mathbb{R}) \cong M_{n\times n}(\mathbb{R})$. Then

$$\begin{aligned} DF_A(B) &= \frac{d}{dt}|_{t=0}\, F(A + tB) \\ &= \frac{d}{dt}|_{t=0}(A + tB)(A + tB)^T \\ &= \frac{d}{dt}|_{t=0} AA^T + t(AB^T + BA^T) + t^2(BB^T) \\ &= AB^T + BA^T. \end{aligned}$$

To see DF_A is surjective onto symmetric matrices, note that we can write any symmetric matrix X as $P + P^T$, where P is upper triangular. (Indeed, take the super diagonal entries of P to be the same as A, and take the diagonal entries of P to be one half times the diagonal entries of A). Then given P and A, we can solve the equation $P = AB^T$ for B, since we are assuming $A \in O(n)$ and so A is invertible because $A^{-1} = A^T$. This shows DF_A is surjective as desired.

(2) Verify that the tangent space at the identity matrix

$$o(n) = \{A \in M_{n\times n}(\mathbb{R}) \,|\, A + A^T = 0\}.$$

Solution: First we claim these two vector spaces have the same dimension. The dimension of the right hand side is $\frac{n^2-n}{2}$, since we are free to specify the upper-triangular part of the matrix A. To compute the dimension of the lefthand side, we can use the regular value theorem. Since $O(n) = F^{-1}(Id)$ with F defined as above, we know $O(n) \subset M_{n\times n}(\mathbb{R})$ is a smooth submanifold with codimension equal to $\dim S(n) = \frac{n^2-n}{2} + n$. Hence $\dim O(n) = \dim o(n) = n^2 - \dim S(n) = \frac{n^2-n}{2}$.

Next, suppose $\gamma(t)$ is a curve in $O(n)$ with $\gamma(0) = Id$, so $\gamma'(0) \in o(n)$. Then

$$F(\gamma(t)) = Id.$$

Differentiating both sides with respect to t and using the chain rule gives

$$DF_{Id}(\gamma'(0)) = 0.$$

This means

$$o(n) \subset \ker DF_{Id} = \{A \in M_{n\times n}(\mathbb{R}) \mid A + A^T = 0\}.$$

Since these two vector spaces have the same dimension, the only way for one of them to be a subset of the other is if they are the same.

(3) Show that the tangent bundle $TO(n)$ can be trivialized, i.e.

$$TO(n) \cong O(n) \times o(n).$$

Solution: Let $(g, v) \in TO(n)$, i.e. $g \in O(n)$ and $v \in T_gO(n)$. Let $L_{g^{-1}}$ denote left multiplication by g^{-1}. This map is a diffeomorphism, so its derivative at g

$$D_gL_{g^{-1}} : T_gO(n) \to T_{Id}O(n) = o(n)$$

is a linear isomorphism. Let

$$\begin{aligned} \Phi : TO(n) &\to O(n) \times o(n) \\ (g, v) &\mapsto (g, D_gL_{g^{-1}}(v)). \end{aligned}$$

Then Φ is a diffeomorphism.