THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Topology

September 4, 2017: Morning Session, 9:00 to 12:00 noon.

1. For a topological space X, define $Sym^2(X) = X \times X/(x, y) \sim (y, x)$, with the quotient topology. Calculate the homology of $Sym^2(S^2)$ (where S^2 is the 2-sphere).

2. Let D be the set of all complex numbers of absolute value ≤ 1, with the induced topology from $\mathbb{C}$. Let n be a natural number. Let $Y = D/\sim$ where $\sim$ is the smallest equivalence relation satisfying

$$e^{it} \sim e^{i(t-(2k+1)\pi/n)}$$

for $t \in \mathbb{R}$, $k\pi/n \leq t \leq (k+1)\pi/n$, $k = 0, \ldots, n-1$, with the quotient topology.

(a) Compute $\pi_1(Y)$.

(b) Classify the compact oriented surface Y.

3. Let X be a connected CW-complex and let $f : Y \to X$ be a (not necessarily connected) covering. Call the covering f *regular* if there exists a (discrete) group G acting on Y such that there exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{Id} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y/G & \xrightarrow{\cong} & X \end{array}$$

where p is the canonical projection, and Y/G has the induced topology. Prove that a covering f as above is regular if and only if the restrictions of f to the connected components of Y are regular coverings (in the ordinary sense), which are isomorphic in the category of (unbased) coverings of X.

4. Consider in $\mathbb{R}^4$ the subspace

$$Z = \{(x, y, z, t) \mid x^2 + y^2 + z^2 = 1,\ t = 0\}.$$

Compute the homology of $\mathbb{R}^4 \smallsetminus Z$ (with the induced topology).

5. Prove that for positive integers n, k, there does not exist a covering $\pi : S^{2n} \to X$ where X is a simplicial complex with $\pi_1(X) \cong \mathbb{Z}/(2k+1)$.

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September 4, 2017: Afternoon Session, 2:00 to 5:00.

(1) (5pt) Show that

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

defines a nonzero deRham cohomology class of $\mathbf{R}^2 - \{(0,0)\}$

(2) (5pt) Any non-constant smooth function of a compact connected manifold has at least two critical points.

(3) (10pt) For each $n \geq 1$, there is a diffeomorphism

$$(TS^n) \times \mathbf{R} \cong S^n \times \mathbf{R}^{n+1}.$$

(4)(15pt) Assuming that every n-dimensional compact manifold M^n can be embedded into some $\mathbf{R}^N$, prove that we can choose $N = 2n+1$. (Hint: Given a nonzero vector $v \neq 0$ in $\mathbf{R}^N$, one can define parallel projection ϕ_v from $\mathbf{R}^N$ to the orthogonal complement of v. If $N > 2n+1$, we can choose some $v \neq 0$ such that the $\phi_v|_{M^n}$ is an embedding).

(5)(15pt) (a) Show that the space of orthogonal matrices

$$O(n) = \{A \in Mat_{n\times n}(\mathbf{R}); \mathbf{A}\mathbf{A}^{\mathbf{t}} = \mathbf{Id}\}$$

is a smooth submanifold of $Mat_{n\times n}(\mathbf{R})$.

(b) Verify that the tangent space at identity matrix

$$o(n) = T_{Id}U(n) = \{A \in Mat_{n\times n}(\mathbf{R}); \mathbf{A} + \mathbf{A}^{\mathbf{t}} = \mathbf{0}\}.$$

(c) Show that the tangent bundle $TO(n)$ can be trivialized, i,e.

$$TO(n) \cong O(n) \times o(n).$$