Topology QR Exam, May 2016, AM

You can choose any **FIVE** out of the six problems below. Indicate clearly on the **FIRST PAGE** of your exam which five you have chosen.

1. Construct infinitely many nonisomorphic compactifications of the open interval $(0, 1)$ which are Hausdorff spaces.

2. Prove that every compact connected 1-dimensional manifold without boundary is diffeomorphic to the unit circle.

3. Let M be a connected non-orientable manifold. Show that its tangent bundle is orientable.

4. Give an example of a connected compact manifold X without boundary satisfying both of the following conditions:

   - $\pi_1(X) \neq 0$.
   - There are no connected covering spaces $f : Y \to X$ with odd degree > 1.

   Explain why your example satisfies both conditions.

5. Let $T = S^1 \times S^1$, viewed as a topological group. Let $f : T \to T$ be a continuous automorphism of T (as a topological group). Assume that the induced linear transformation $H_1(f) : H_1(T, \mathbb{Z}) \to H_1(T, \mathbb{Z})$ has an even trace. Show that f has at least one fixed point other than 0.

6. Let M be a compact manifold without boundary.

   (a) Show that there is no submersion $M \to \mathbb{R}$.

   (b) If M is simply connected, show that any submersion $M \to \mathbb{RP}^2$ has disconnected fibers.

Topology QR Exam, May 2016, PM

You can choose any **FIVE** out of the six problems below. Indicate clearly on the **FIRST PAGE** of your exam which five you have chosen.

1. Give the definition of a proper action of a group on a topological space, and construct an example of a group action of Γ on a Hausdorff topological space X such that the quotient $\Gamma\backslash X$ is not a Hausdorff space.

2. Let G be a connected Lie group, and $H = \tilde{G}$ be the universal covering space of G. Show that H has a natural Lie group structure such that the projection map $H \to G$ is a Lie group homeomorphism.

3. Let M be a compact smooth manifold with nonempty boundary ∂M. Show that there is no smooth deformation retraction $M \to \partial M$.

4. Let X be the complement of a point in the torus $S^1 \times S^1$.

   (a) Calculate $\pi_1(X)$.

   (b) Show that every map $\mathbb{RP}^n \to X$ is null-homotopic for $n \geq 2$.

5. Show that the canonical map $\mathbb{C}^{n+1} - \{0\} \to \mathbb{CP}^n$, given by sending a point $x \in \mathbb{C}^{n+1} - \{0\}$ to the line $\ell_x \subset \mathbb{C}^{n+1}$ connecting x to 0, does not have a section.

6. Assume that X is a connected finite CW complex, and that the universal cover $\tilde{X}$ of X is compact. Show that $\tilde{X}$ cannot be contractible unless X is itself contractible.