Applied Functional Analysis Qualifying exam

Assigned January 9, 2022

Problem 1

Let H_1 and H_2 be Hilbert spaces and $A : H_1 \mapsto H_2$ a compact operator. Prove that adjoint $A^\star$ of A is compact.

Problem 2

Let a, b be two real numbers satisfying $a < b$ and consider the space $L^2([a, b])$, with the usual inner product

$$\langle f, g \rangle = \int_a^b \overline{f(x)} g(x) dx, \qquad \forall\, f, g \in L^2([a, b]),$$

where the bar denotes complex conjugate. Introduce also the space $H := L^2([a, b] \times [a, b])$ and denote its inner product by

$$\langle F, G \rangle_H = \int_a^b \int_a^b \overline{F(x, y)} G(x, y) dx dy, \qquad \forall\, F, G \in H.$$

1. Let $(\varphi_n(x))_{n\geq 1}$ and $(\tilde{\varphi}_n(x))_{n\geq 1}$ be two orthonormal bases of $L^2([a, b])$. Prove that $(\varphi_n(x)\tilde{\varphi}_m(y))_{n,m\geq 1}$ is an orthonormal basis of H.

2. Consider the compact linear integral operator $A : L^2([a, b]) \mapsto L^2([a, b])$ defined by

   $$Af(x) := \int_a^b e^{-(x-y)^2} f(y) dy.$$

   Denote its eigenvalues by λ_n and its eigenfunctions by $u_n(x)$, for $n \geq 1$. Prove that the kernel of A satisfies

   $$e^{-(x-y)^2} = \sum_{n=1}^{\infty} \lambda_n u_n(x) \overline{u_n(y)},$$

   where the series converges in the H norm. Prove also that

   $$\int_a^b \int_a^b e^{-2(x-y)^2} dx dy = \sum_{n=1}^{\infty} \lambda_n^2.$$

Problem 3

Let H be a Hilbert space over the complex field $\mathbb{C}$.

1. Consider a sequence $(x_n)_{n\geq 1}$ in H that is weakly Cauchy. This means that for any linear bounded functional $\varphi \in H^\star$, the sequence $\big(\varphi(x_n)\big)_{n\geq 1}$ is Cauchy in $\mathbb{C}$. Define the sequence of linear maps

   $$F_n : H^\star \mapsto \mathbb{C}, \quad F_n(\varphi) := \varphi(x_n), \qquad \forall\, \varphi \in H^\star, \;\; \forall\, n \geq 1$$

   and use it to prove that $(x_n)_{n\geq 1}$ is a bounded sequence.

2. Let S be a set in H with the following property: Every non-empty subset of S has a weak Cauchy sequence. Prove that S is bounded.

Problem 4

Let δ_y denote the Dirac delta located at $y \in \mathbb{R}$, and let $\delta_y^{(n)}$ denote its n^{th} distributional derivative.

1. Does $\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}$ define a distribution in $\mathscr{D}'(\mathbb{R})$? Prove or disprove.
2. Does $\sum_{n=1}^{\infty} \delta_{1/n}^{(n)}$ define a distribution in $\mathscr{D}'((0,\infty))$? Prove or disprove.
3. Let $f(x)$ be a $C^{\infty}(\mathbb{R})$ function with $f(0) = 0$ and let $F \in \mathscr{D}'(\mathbb{R})$ be a distribution with support $\{0\}$. Is $f(x)F$ the zero distribution? Prove or disprove.

Problem 5

Suppose that A is a compact self-adjoint operator on a Hilbert space H, and suppose that the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of A are all nonzero and satisfy $|\lambda_n| \le Cn^{-1}$ for some $C > 0$. Let $\{e_n\}_{n=1}^{\infty}$ denote the corresponding orthonormal eigenvectors.

1. Show that $f := \sum_{n=1}^{\infty} n^{-1/2} \lambda_n e_n \in H$.
2. Is $f \in \operatorname{Ran}(A)$? Prove or disprove.