Applied Functional Analysis QR Exam

May 2023

Problem 1: Let the integral operator $T : C([0,1]) \to C([0,1])$ be given by

$$Tf(x) = \int_0^x f(y)\,dy\,.$$

Show that the spectrum of T consists of the single point 0.

Problem 2: Express the Fourier transform of the tempered distribution defined by the function $f : \mathbb{R} \to \mathbb{R}, f(x) = |x|$ as a principal value distribution (i.e. for a test function $\phi \in \mathcal{S}(\mathbb{R})$, express $\widehat{|x|}\,[\phi]$ as a principal value integral in ϕ).

Problem 3: Show that the function $f(x) = x^2 - 3x + 2 - e^x$ has a unique root in the interval $[0,1]$. Show that it can be approximated with the iterative scheme

$$x_{n+1} = \frac{1}{3}\left(2 - e^{x_n} + {x_n}^2\right)$$

provided that $x_0 \in [0,1]$. Estimate (find an upper bound on) the number of iterations required to ensure that the error is less than 10^{-6}. The bound need not be optimal, but be reasonable.

Problem 4: Let $\{x^n\}_{n=1}^{\infty}$ be a sequence in ℓ^1. (Note on notation: This means each x^n is an absolutely summable infinite sequence: $x^n = \{x_1^n, x_2^n, x_3^n, \ldots\}$). Prove that $\{x^n\}_{n=1}^{\infty}$ is weakly convergent if and only if it is strongly convergent.

Note: A sequence $\{y^n\}_{n=1}^{\infty} \subset \ell^1$ converges weakly to $y^* \in \ell^1$ if $\lim_{n\to\infty} \sum_{j=1}^{\infty} y_j^n \phi_j = \sum_{j=1}^{\infty} y_j^* \phi_j$ for every $\phi \in \ell^\infty$.

Problem 5: Let X be a Hilbert space, $T : X \to X$ a bounded linear map. Show that T takes every weakly convergent sequence to a strongly convergent sequence if and only if T is compact. Give a counterexample when X is just a Banach space.