Applied Functional Analysis QR Exam - January 2023

Problem 1 Let $A : X \mapsto Y$ be a linear operator between normed vector spaces X and Y, with the following property: For any sequence $(x_n)_{n\geq 1}$ in X converging to 0, the sequence $(Ax_n)_{n\geq 1}$ is bounded. Prove that A is continuous.

Problem 2 The secant method is an iterative method for solving an equation $f(x) = 0$, for a function $f : \mathbb{R} \mapsto \mathbb{R}$. Starting with two points $x_0, x_1 \in \mathbb{R}$, it computes the sequence $(x_n)_{n\geq 1}$ as follows:

$$x_{n+1} = x_n - f(x_n)\left[\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right], \qquad n \geq 1.$$

Prove that the secant method applied to the function $f(x) = x^2 - 2$ and with starting points $x_0, x_1 > \sqrt{2}$ converges to a root.

Problem 3 Consider the linear operator $T : \ell^2 \mapsto \ell^2$, defined by $T(x_1, x_2, x_3, \ldots) = \left(\frac{1}{1}x_1, \frac{1}{\sqrt{2}}x_2, \frac{1}{\sqrt{3}}x_3, \ldots\right)$, for all $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ in ℓ^2. Prove that T is a compact linear operator. Is the range of T closed? Prove or disprove it.

Problem 4 This problem has two parts:

1. Let A be a bounded linear operator $A : H \mapsto H$, where H is a Hilbert space. Prove that if λ belongs to the residual spectrum of A, then $\bar{\lambda}$, the complex conjugate of λ, belongs to the point spectrum of the adjoint $A^\star$ of A.
2. Consider the right and left shift operators defined on the Hilbert space ℓ^2 of sequences $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ with complex terms, satisfying $\sum_{j=1}^{\infty} |x_j|^2 < \infty$,

$$R : \ell^2 \mapsto \ell^2, \quad R((x_1, x_2, \ldots)) = (0, x_1, x_2, \ldots)$$
$$L : \ell^2 \mapsto \ell^2, \quad L((x_1, x_2, \ldots)) = (x_2, x_3, \ldots).$$

   Prove that

   (a) The resolvent sets of R and L are both the exterior of the unit disk $\{\lambda \in \mathbb{C} |\ |\lambda| > 1\}$.

   (b) Determine the point spectrum, the continuous spectrum and the residual spectrum of R and L. *Hint: Part 1 of this problem should be useful.*

Problem 5 Show that the distributional derivative of $\log|x| : \mathbb{R} \mapsto \mathbb{R}$ is the principal value distribution p.v.$1/x$.