Numerical Analysis Qualifying Review

May, 2021

1. Consider the initial-value problem for $y(t)$:

$$\begin{gathered} y' = \frac{t}{y} \\ y(0) = 2 \\ 0 \leq t \leq 5 \end{gathered}$$

   (a) Solve the initial-value problem exactly.

   (b) Is the problem well-posed? Justify.

   (c) Approximate the solution on $[0, 2]$ using the Euler method with step size $\Delta t = 1$. Put $w_0 = y(0)$ and approximate $w_i \approx y(i)$.

   (d) Sketch $f(t, y) = \frac{t}{y}$ on the t-y plane by putting an arrow at (t, y) with $\Delta t = 1$ and $\Delta y = f(t, y)$, at representative and useful points (t, y). Sketch the exact solution, asymptotes, and the numeric solution to the IVP on $[0, 2]$. Indicate geometrically why the problem is or is not well-posed.

Solution

(a) Separate: $y\mathrm{d}y = t\mathrm{d}t$. Integrate: $y^2 = t^2 + c$. By the intitial condition, $c = 4$, so $y = \pm\sqrt{t^2 + 4}$. Using the initial condition again, choose the positive branch.

(b) It is well-posed, because $f(t, y)$ satisfies a Lipschitz condition in y in a domain that includes the solution (which is bounded away from $y = 0$). Without solving the equation, we can see that $y(0) = 2 > 0$ and $y' > 0$, so, $y \geq 2 > 0$. As $y \to \infty$, we have $y' \to 0$, whence the Lipschitz condition.

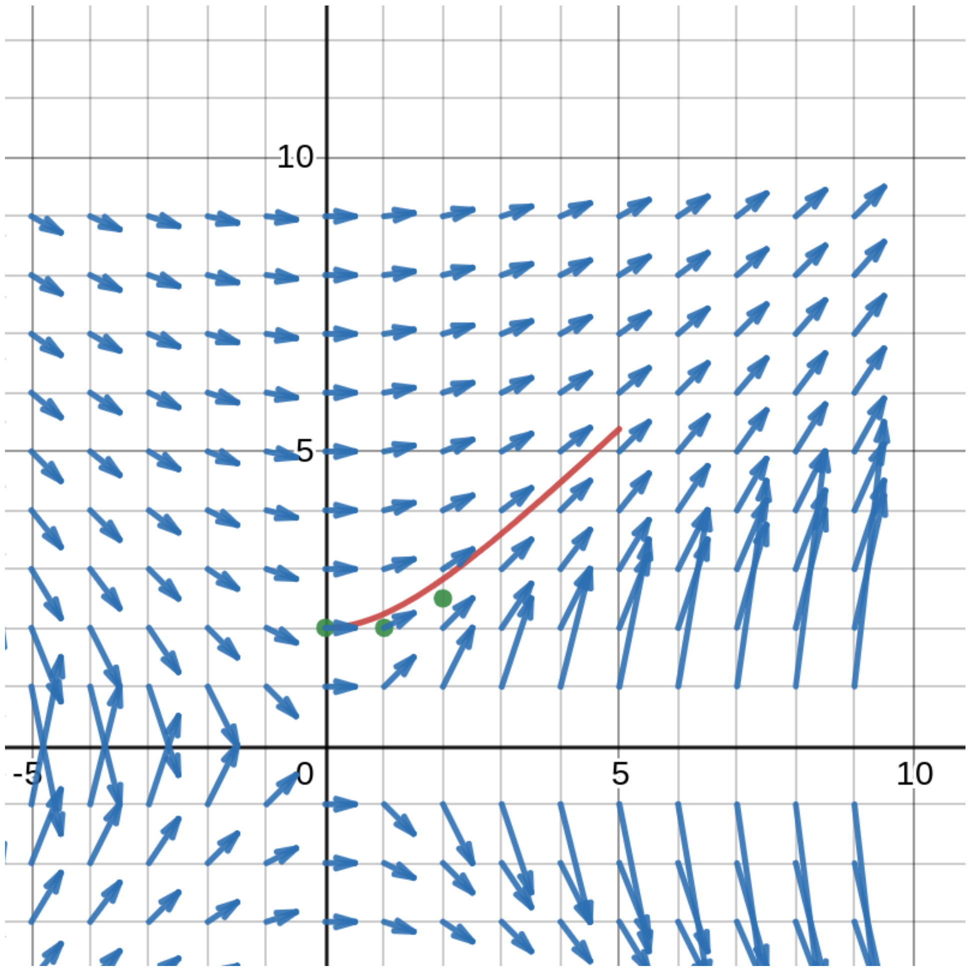
(c) $w_0 = y(0) = 2$. Then

n	w_n
0	2
1	$w_1 = w_0 + \Delta t \cdot f(0, w_0) = 2 + 1 \cdot 0 = 2$
2	$w_2 = w_1 + \Delta t \cdot f(1, w_1) = 2 + 1 \cdot \frac{1}{2} = \frac{5}{2} = 2.5$

(d) If we hold t constant (move up and down at a vertical line), we see arrows that tend to converge or, anyway, do not diverge badly. More precisely, they do so in a neighborhood of the solution large enough to contain the approximation and, in any case, stays in the first quadrant. That is the geometric view of the Lipschitz condition.

The approximation is three points, in green. Interpolating them is optional, and should come with justification of the interpolation method: one polynomial, linear splines, etc.

The solution is a hyperbola with asymptote $y = x$ omitted from the graph to reduce clutter.



2. The modified Euler method for the initial-value problem $y'(t) = f(t, y); y(a) = \alpha$ on $t \in [a, b]$ is as follows:

$$\begin{aligned} \widetilde{w} &= w_i + \tfrac{h}{2} f(t_i, w_i) \\ w_{i+1} &= w_i + h f(t_i + \tfrac{h}{2}, \widetilde{w}) \end{aligned}$$

Derive the local truncation error, for "reasonable" functions f. (Answer in the form $O(h^k)$.) Give appropriate conditions on f to achieve that order, and explain.

Solution

The local truncation error is, assuming $w_i = y_i$,

$$|y_{i+1} - w_{i+1}| = O(h^3).$$

Existence of all first and second partial derivatives of f suffices. This is derived from Taylor series expansions. Put $t^* = t_i + h/2$. By assumption,

$$\widetilde{w} = y_i + \frac{h}{2} f(t_i, y_i).$$

Then $y(t^*) = y_i + \frac{h}{2} f(t_i, y_i) + O(h^2)$, so $\widetilde{w} = y(t^*) \pm O(h^2)$. Since (we'll assume) f satisfies a Lifschitz condition in y, we have

$$h f\left(t^*, \widetilde{w}\right) = h f\left(t^*, y\left(t^*\right)\right) \pm O(h^3).$$

Thus $w_{i+1} = y_i + hf(t^*, y(t^*)) + O(h^3)$. The next calculation amounts to midpoint integration, which increases the order by a factor of h. That is,

$$\begin{aligned} y(t_{i+1}) - y(t_i) &= \int_{t_i}^{t_i+h} f(t, y(t))(d)t \\ &= \int_{t_i}^{t_i+h} f(t^*, y(t^*)) + f'(t^*, y(t^*))(t - t^*) + O(t - t^*)^2 \mathrm{d}t \\ &= hf(t^*, y(t^*) + O(h^3), \end{aligned}$$

by odd symmetry of $(t - t^*)$ about t^*.

Thus $|y_{i+1} - w_{i+1}| \leq O(h^3)$ and $\left| \frac{y_{i+1}-y_i}{h} - \frac{w_{i+1}-y_i}{h} \right| \leq O(h^2)$

3. Consider the boundary value problem

$$-u'' + \pi^2 u = 2\pi^2 \sin(\pi x)$$
$$u(0) = u(1) = 0.$$

   (a) Set up a finite difference approximation with $h = \Delta x = \frac{1}{4}$ as a system of equations in unknowns $w_1 \approx u(h), w_2 \approx u(2h), w_3 \approx u(3h)$. Use the central form of the second difference. (You can include "unknowns" w_0 and w_4 if that makes it cleaner to set up the system.)

   (b) Set up a Jacobi iteration to solve this system. (No need to solve by hand.)

   (c) Set up a Gauss-Seidel iteration.

   (d) Comment on the convergence and efficiency of the iterative schemes.

   (e) What properties of the iterative schemes or their convergence would be different if the term $\pi^2 u$ were instead $-32u$? What if instead of $\pi^2 u$ it were $-\pi^2 u$?

Solution

Bradie pp. 665 ff.

(a)

$$\begin{bmatrix} 1 & 0 & & & \\ -1 & 2 + (\pi/4)^2 & -1 & & \\ & -1 & 2 + (\pi/4)^2 & -1 & \\ & & -1 & 2 + (\pi/4)^2 & -1 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2}(\pi/4)^2 \\ 2(\pi/4)^2 \\ \sqrt{2}(\pi/4)^2 \\ 0 \end{bmatrix}.$$

(b) This kind of low-bandwidth matrix is well-suited to Jacobi iteration. Write the matrix above as $L + D + U$ as a sum of the lower triangle, diagonal, and upper triangle, and let $\vec{w}$ denote the vector of unknowns and $\vec{b}$ denote the right hand side. Then Jacobi iteration is given by

$$w^{(k+1)} = -D^{-1}(L + U)w^{(k)} + D^{-1}\vec{b}.$$

(c) With notation as above, Gauss-Seidel is given by

$$w^{(k+1)} = -(D + L)^{-1}Uw^{(k)} + (D + L)^{-1}\vec{b}.$$

(d) The original matrix $(L + D + U)$ is strictly diagonally dominant (diagonal entry is larger in magnitude than the sum of magnitudes of other elements in the row), so both iterative schemes converge. The norm of the n'th error vector $\vec{e}^{(n)}$ decreases with bound $\|D^{-1}(L + U)\|^n \cdot \|e^{(0)}\|$ (Jacobi) or $\|(D + L)^{-1}U\|^n \cdot \|e^{(0)}\|$ (Gauss-Seidel). Each iteration requires time $O(n)$ for a tridiagonal $n \times n$ matrix.

(e) If we apply the above to $-u'' - 32u = *$ with Δx still equal to $1/4$, we get the matrix

$$\begin{bmatrix} 1 & 0 & & & \\ -1 & 0 & -1 & & \\ & -1 & 0 & -1 & \\ & & -1 & 0 & -1 \\ & & & 0 & 1 \end{bmatrix}$$

The corresponding matrix D is not invertible and neither is $(L+D)$, so neither Jacobi nor Gauss-Seidel will work as written. The homogeneous system still has a well-behaved solution, $ce^{\sqrt{32}t}$. We can restore Jacobi and Gauss-Seidel by making Δx smaller.

If we have $-u'' - \pi^2 u = 2\pi^2 \sin(\pi x)$, then...

4. Consider the initial boundary value problem

$$u_t = \tfrac{1}{16} u_{xx}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = 2\sin(2\pi x)$$

(a) Sketch the domain on the x-t plane with x horizontal and t vertical.

(b) Set up a continuous-time and central-space discretization

$$\frac{d\vec{v}}{dt} = -\frac{D}{(\Delta x)^2}\left[A\vec{v}(t) + \vec{b}(t)\right].$$

That is, find the scalar D, the matrix A, and the vector $\vec{b}$. (The vector $\vec{v}(t)$ approximates the solution $\vec{u}(t)$ discretized in space at time t.) Use a central difference for the discretization of u_{xx}.

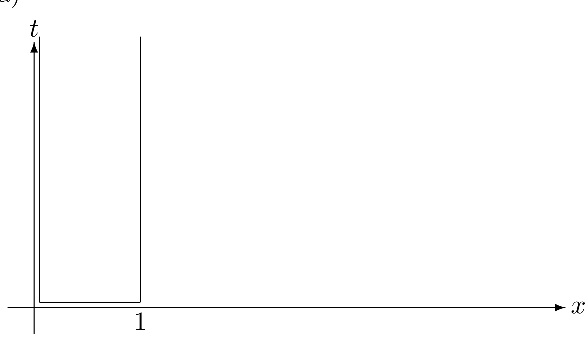
(c) Now discretize your system in time to first order, using $\vec{w}^{(n)}$ as unknown vectors and n corresponding to discrete time. Use the forward-time discretization, which result in an overall Forward-Time, Central-Space scheme. Derive the evolution $w^{(n+1)} = (I - \lambda A)w^{(n)} - \lambda b^{(n+1)}$, where the superscript indicates discrete time. Comment on the value of λ and its role in convergence.

(d) Explicitly solve for $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^{(n)}$ (omitting $w_0 = w_4 = 0$), using $\Delta x = \Delta t = \frac{1}{4}$. Write the values of $\vec{w}^{(n)}$ for $n = 0, 1, 2$.

Solution

Bradie pp. 808 ff.

(a)



(b) We have $D = \frac{1}{16}$, $\vec{b}(t) = \begin{bmatrix} -u(0,t) & 0 & 0 & \dots & 0 & -u(1,t) \end{bmatrix}^{\mathrm{T}}$, and

$$A = \begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}.$$

(c) We can discretize time to first order, getting

$$\frac{\vec{w}^{(n+1)} - \vec{w}^{(n)}}{\Delta t} = -\frac{D}{\Delta x^2}\left[A\vec{w}^{(n)} + b^{(n)}\right].$$

This leads to $\vec{w}^{(n+1)} = (I - \lambda A)w^{(n)} - \lambda\vec{b}^{(n+1)}$, as desired. We must have $(I + \lambda A)$ invertible, which gives a condition on λ as $\lambda < \|A\|^{-1}$. The norm of A is constant, so $\lambda = D\Delta t/(\Delta x)^2$ must be bounded above by a constant, and so Δt is bounded above by a constant times $(\Delta x)^2/D$.

(d) For the specified Δx and Δt, we get $\lambda = D\Delta t/(\Delta x)^2 = 1/4$ and we get

$$I - \lambda A = \begin{bmatrix} .5 & .25 & 0 \\ .25 & .5 & .25 \\ 0 & .25 & .5 \end{bmatrix},$$

in the unknown $\vec{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}^{\mathrm{T}}$, omitting $w_0 = w_4 = 0$. The right hand side $\vec{b}^{(n)}$ is zero for all n. We get:

$$w_0 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \to w_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \to w_2 = \begin{bmatrix} .5 \\ 0 \\ -.5 \end{bmatrix}$$

5. Consider the wave equation

$$\begin{aligned} u_{tt} &= \tfrac{1}{25}u_{xx} \\ 0 < x &< 1 \\ 0 < t &< \infty \\ u(0,t) &= -\sin(t/5) \\ u(1,t) &= \sin(1 - t/5) \\ u(x,0) &= \sin x \\ u_t(x,0) &= -\tfrac{1}{5}\cos x. \end{aligned}$$

We will set up a discretization $w_j^{(n)} \approx (u(j\Delta x, n\Delta t)$ by

$$\frac{w_j^{(n+1)} - 2w_j^{(n)} + w_j^{(n-1)}}{(\Delta t)^2} = \frac{1}{25}\frac{w_{j-1}^{(n)} - 2w_j^{(n)} + w_{j+1}^{(n)}}{(\Delta x)^2}.$$

(a) For given Δx, give bounds on Δt to make the discretization stable in ℓ_2.

(b) Explain how to compute $w_j^{(1)}$ as a special case.

Solution

Bradie, pp. 924ff.

(a) Solving the discretization above for $w_j^{(n+1)}$ gives

$$w_j^{(n+1)} = \lambda w_{j-1}^{(n)} + 2(1-\lambda)w_j^{(n)} + \lambda w_{j+1}^{(n)} - w_j^{(n-1)},$$

where $c = 1/5$ and $\lambda = (c\Delta t/\Delta x)^2$. Plugging in $w_j^{(n)} = r^n e^{i(j\theta)}$, we get

$$r^{n+1}e^{ij(\theta)} = r^n e^i j(\theta)[\lambda e^{-i\theta} + 2(1-\lambda) + \lambda e^{i\theta}] - r^{n-1}e^{ij(\theta)},$$

which reduces to

$$r^2 - 2[(1-\lambda) + \lambda\cos\theta] + 1 = 0.$$

For the method to be stable, both roots must be less than or equal to 1 in magnitude. Since the constant term is 1, the product of the roots is 1, so we can't have distinct real roots. That means the discriminant $4[(1-\lambda)+\lambda\cos\theta]^2 - 4$ must be 0 or negative,

$$-1 \le (1-\lambda) + \lambda\cos\theta \le 1.$$

The inequality on the right holds for all λ but the inequality on the left requires $\lambda \le 1$. If $\lambda = 1$, there's a double root of $+1$ or -1. If $\lambda < 0$, then the roots are complex conjugates with the same magnitude and product 1, so each root has magnitude 1. We conclude that $\lambda \le 1$, so that $\Delta t \le \Delta x/c$.

(b) For $n = 1$, we get

$$w_j^{(1)} = \lambda w_{j-1}^{(0)} + 2(1-\lambda)w_j^{(0)} + \lambda w_{j+1}^{(0)} - w_j^{(-1)}. \quad (1)$$

We can use $u_t(x,0) = -\frac{1}{5}\cos x = g_j$ to impute a value for $w_j^{(-1)}$, where j is the discretization index corresponding to x. We have

$$\frac{w_j^{(1)} - w_j^{(-1)}}{2\Delta t} \approx g_j,$$

so we can solve for $w_j^{(-1)}$ in terms of $w_j^{(1)}$ and other known quantities, and rewrite (1) as

$$w_j^{(1)} = \frac{1}{2}\lambda w_{j-1}^{(0)} + (1-\lambda)w_j^{(0)} + \frac{1}{2}\lambda w_{j+1}^{(0)} + \Delta t g_j.$$

The initial conditions give us $w_{j\pm1}^{(0)}$.