Numerical Analysis Qualifying Review

May, 2021

1. Consider the initial-value problem for $y(t)$:

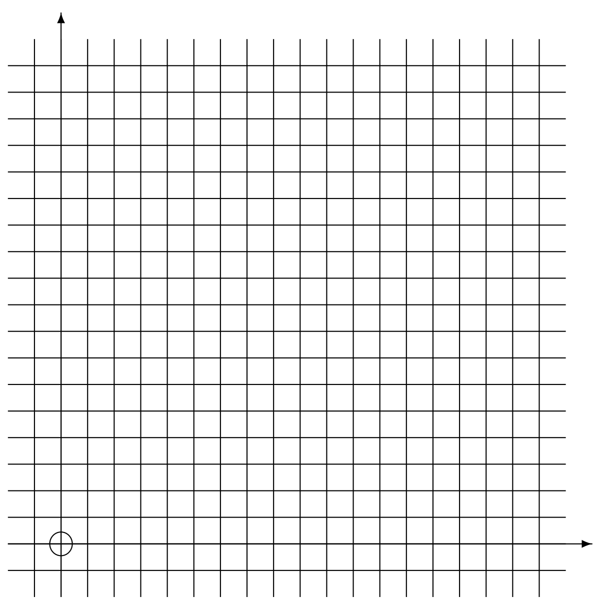
$$
\begin{gathered}
y' = \frac{t}{y} \\
y(0) = 2 \\
0 \le t \le 5
\end{gathered}
$$

   (a) Solve the initial-value problem exactly.

   (b) Is the problem well-posed? Justify.

   (c) Approximate the solution on $[0, 2]$ using the Euler method with step size $\Delta t = 1$. Put $w_0 = y(0)$ and approximate $w_i \approx y(i)$.

   (d) Sketch $f(t, y) = \frac{t}{y}$ on the t-y plane by putting an arrow at (t, y) with $\Delta t = 1$ and $\Delta y = f(t, y)$, at representative and useful points (t, y). Sketch the exact solution, asymptotes, and the numeric solution to the IVP on $[0, 2]$. Indicate geometrically why the problem is or is not well-posed.



2. The modified Euler method for the initial-value problem $y'(t) = f(t, y); y(a) = \alpha$ on $t \in [a, b]$ is as follows:

$$\begin{aligned} \widetilde{w} &= w_i + \tfrac{h}{2} f(t_i, w_i) \\ w_{i+1} &= w_i + h f(t_i + \tfrac{h}{2}, \widetilde{w}) \end{aligned}$$

Derive the local truncation error, for "reasonable" functions f. (Answer in the form $O(h^k)$.) Give appropriate conditions on f to achieve that order, and explain.

3. Consider the boundary value problem

$$\begin{gathered} -u'' + \pi^2 u = 2\pi^2 \sin(\pi x) \\ u(0) = u(1) = 0. \end{gathered}$$

   (a) Set up a finite difference approximation with $h = \Delta x = \frac{1}{4}$ as a system of equations in unknowns $w_1 \approx u(h), w_2 \approx u(2h), w_3 \approx u(3h)$. Use the central form of the second difference. (You can include "unknowns" w_0 and w_4 if that makes it cleaner to set up the system.)

   (b) Set up a Jacobi iteration to solve this system. (No need to solve by hand.)

   (c) Set up a Gauss-Seidel iteration.

   (d) Comment on the convergence and efficiency of the iterative schemes.

   (e) What properties of the iterative schemes or their convergence would be different if the term $\pi^2 u$ were instead $-32u$? What if instead of $\pi^2 u$ it were $-\pi^2 u$?

4. Consider the initial boundary value problem

$$u_t = \tfrac{1}{16} u_{xx}$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = 2\sin(2\pi x)$$

   (a) Sketch the domain on the x-t plane with x horizontal and t vertical.

   (b) Set up a continuous-time and central-space discretization

$$\frac{d\vec{v}}{dt} = -\frac{D}{(\Delta x)^2}\left[A\vec{v}(t) + \vec{b}(t)\right].$$

   That is, find the scalar D, the matrix A, and the vector $\vec{b}$. (The vector $\vec{v}(t)$ approximates the solution $\vec{u}(t)$ discretized in space at time t.) Use a central difference for the discretization of u_{xx}.

   (c) Now discretize your system in time to first order, using $\vec{w}^{(n)}$ as unknown vectors and n corresponding to discrete time. Use the forward-time discretization, which result in an overall Forward-Time, Central-Space scheme. Derive the evolution $w^{(n+1)} = (I - \lambda A)w^{(n)} - \lambda b^{(n+1)}$, where the superscript indicates discrete time. Comment on the value of λ and its role in convergence.

   (d) Explicitly solve for $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}^{(n)}$ (omitting $w_0 = w_4 = 0$), using $\Delta x = \Delta t = \frac{1}{4}$. Write the values of $\vec{w}^{(n)}$ for $n = 0, 1, 2$.

5. Consider the wave equation

$$\begin{gathered} u_{tt} = \tfrac{1}{25} u_{xx} \\ 0 < x < 1 \\ 0 < t < \infty \\ u(0,t) = -\sin(t/5) \\ u(1,t) = \sin(1 - t/5) \\ u(x,0) = \sin x \\ u_t(x,0) = -\tfrac{1}{5}\cos x. \end{gathered}$$

   We will set up a discretization $w_j^{(n)} \approx (u(j\Delta x, n\Delta t)$ by

$$\frac{w_j^{(n+1)} - 2w_j^{(n)} + w_j^{(n-1)}}{(\Delta t)^2} = \frac{1}{25}\frac{w_{j-1}^{(n)} - 2w_j^{(n)} + w_{j+1}^{(n)}}{(\Delta x)^2}.$$

   (a) For given Δx, give bounds on Δt to make the discretization stable in ℓ_2.

   (b) Explain how to compute $w_j^{(1)}$ as a special case.