Applied Functional Analysis QR Exam - August 2021

Problem 1 Consider the sequence of integrable functions $(f_n)_{n\geq 1}$, where

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n(x) = -\frac{2n^{3/2}x}{\pi(1+nx)^2}, \qquad n = 1, 2, \ldots. \tag{1}$$

1. Prove that the sequence $(f_n)_{n\geq 1}$ converges in the sense of distributions to $\delta'(x)$, the derivative of the Dirac delta distribution.
2. Does the sequence $(f_n)_{n\geq 1}$ converge pointwise? Does it converge uniformly?

Problem 2

1. Let $g : X \mapsto X$ be a mapping of a Banach space X into itself. Suppose that there exists a closed ball $\overline{B(x_0, R)}$ contained in X, centered at x_o and of radius R, where g satisfies

$$\|g(x) - g(y)\| \leq C\|x - y\|, \qquad \forall\, x, y \in \overline{B(x_0, R)}, \tag{2}$$

for a constant $C \in (0, 1)$. Suppose also that

$$\|g(x_o) - x_o\| < (1 - C)R. \tag{3}$$

Prove that the sequence $(x_n)_{n\geq 1}$ in X, defined by $x_n = g(x_{n-1})$ for all $n \geq 1$, converges to a point $x \in \overline{B(x_0, R)}$, which is the unique fixed point of $g(x)$ in the closed ball.

2. Use the result above to set up an iteration for finding a root of the polynomial $x^3 - 4x - 1$.

Problem 3 Consider the linear operator $L : \ell^2 \mapsto \ell^2$ defined on the Hilbert space of square summable sequences by

$$Lx = \left(\frac{\xi_j}{\sqrt{j}}\right)_{j\geq 1}, \qquad \forall\, x = (\xi_j)_{j\geq 1} \in \ell^2. \tag{4}$$

Find the point spectrum, the continuous spectrum and the residual spectrum of L.

Problem 4 Let A be a bounded linear operator defined on a Hilbert space $\mathcal{H}$, and let $B : \mathcal{H} \to \mathcal{H}$, $B := \mathbb{I} + A^*A$, where A^* denotes the adjoint of A.

1. Show that $\operatorname{ran} B$ is closed.
2. Is B injective? Prove or find a counterexample.
3. Is B onto? Again prove or find a counterexample.
4. Can A fail to have a closed range? Prove or find a counterexample.

Problem 5 Determine whether for every $f \in L^2(\mathbb{R})$ there is a unique solution $u \in L^2(\mathbb{R})$ of

$$u(x) + \int_{\mathbb{R}} \mathrm{e}^{-\frac{1}{2}(x-y)^2} u(y)\,\mathrm{d}y = f(x),$$

and prove it. If the answer is in the affirmative, give a formula for $u(x)$ in terms of f.