QUALIFYING REVIEW EXAM: APPLIED FUNCTIONAL ANALYSIS
AUGUST 2020

(1) Consider the nonlinear integral equation

$$u(x) = \lambda \int_0^1 \frac{\mathrm{d}y}{1 + x + u(y)}, \quad 0 \le x \le 1,$$

where λ is a real parameter and $u(x)$ is to be found. Prove that if $0 \le \lambda < 1$, then there exists exactly one solution that is continuous on the interval $[0, 1]$ and satisfies $u(x) \ge 0$.

(2) Consider the linear operator A defined on the Banach space $L^1(-1, 1)$ with dense domain $C([-1, 1])$ and action $Au(x) := u(x^2)$. Such operators arise in the representation theory of nonlinear maps $f : (-1, 1) \to (-1, 1)$; in this case, $f(x) := x^2$.
  (a) Find the kernel $\ker(A)$ and range $\operatorname{ran}(A)$.
  (b) Is A a bounded operator (with respect to the $L^1(-1, 1) \to L^1(-1, 1)$ norm)? Prove or disprove.
  (c) Does A have any nonzero eigenvalues? Prove or disprove.

(3) For each N, the set of monomials $\{x^n\}_{n=0}^N$ is linearly independent in $\mathcal{H} = L^2(-1, 1)$. The *Legendre polynomials* $\{p_n(x)\}_{n=0}^\infty$ are the result of applying the Gram-Schmidt process to $\{x^n\}_{n=0}^\infty$.
  (a) Find $p_0(x)$, $p_1(x)$, and $p_2(x)$.
  (b) Find the element in the closed linear subspace of $\mathcal{H}$ spanned by 1, x, and x^2 that minimizes the distance to x^3.

(4) Recall that the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R})$ is given by

$$\hat{\varphi}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) \mathrm{e}^{-\mathrm{i}kx} \, \mathrm{d}x.$$

For each of the following tempered distributions, (i) find its Fourier transform and express its action on a test function φ directly, i.e., not in terms of $\hat{\varphi}$, and (ii) state whether the resulting distribution is regular or singular, and if regular identify the function represented.
  (a) $\delta'(x)$ (derivative of the delta distribution).
  (b) $\operatorname{sgn}(x)$ (regular distribution given by a function)

(5) Consider the bounded Fredholm operator A defined on the Hilbert space $\mathcal{H} = L^2(0, 1)$ by

$$Au(x) := \int_x^1 u(y) \, \mathrm{d}y.$$

  (a) Find the adjoint.
  (b) Use part (a) to determine for which functions $f \in \mathcal{H}$ there is a solution $u(x)$ of the equation $Au + u = f$.
  (c) Suppose that $u_n \rightharpoonup u$ ("$\rightharpoonup$" denotes weak convergence). Using only the boundedness of A, prove that $Au_n \rightharpoonup Au$.