QUALIFYING REVIEW EXAM: APPLIED ANALYSIS
SECOND PART, MAY 2019

(1) Consider the differential equation $y' = \mathrm{i}\mu y$ for $y = y(t)$ with initial condition $y(0) = y_0 \in \mathbb{C}$ and parameter $\mu \in \mathbb{R}$. Let u_n be the approximation of $y(nh)$ obtained after n steps of the trapezoidal scheme with step size $h > 0$ and initial condition $u_0 = y_0$.
 (a) Show that $|u_n| = |y(nh)|$ for all n, $h > 0$, and $\mu \in \mathbb{R}$.
 (b) Find the order of accuracy for the phase of the numerical solution; in other words find $C \neq 0$ and $r > 0$ such that $\arg(u_n) - \arg(y(nh)) = Ch^r + O(h^{r+1})$ as $h \to 0$ when $nh = t$ is fixed.

(2) For the initial-value problem $y' = f(y)$ with initial condition $y(0) = y_0$, consider the family of numerical schemes parametrized by $\theta \in [0, 1]$:

$$u_{n+1} = u_n + h\left[(1-\theta)f(u_{n+1}) + \theta f(u_n)\right], \quad u_0 = y_0.$$

 (a) By analyzing the local truncation error, determine how the order of the method depends on θ.
 (b) Show that the method is A-stable if and only if $0 \le \theta \le \frac{1}{2}$.
 (c) If the method is applied in the special case that $f(y) := \lambda y$, show that given h there exists a value of θ such that the numerical solution agrees with the exact solution of the initial-value problem.

(3) Consider solving the equation $y''(x) = f(x)$ on $[0, 1]$ with $f : [0, 1] \to \mathbb{R}$ given and under the mixed Dirichlet-Neumann boundary conditions $y'(0) = y(1) = 0$. Taking equally-spaced nodes $x_j = jh$, $j = 0, \ldots, N$ (so $h = 1/N$), discretize the second derivative using a standard second difference scheme D_+D_- and for the numerical approximation u_j of $y(x_j)$ implement the Dirichlet boundary condition by setting $u_N = 0$ and the Neumann boundary condition by introducing a fictitious node x_{-1} with corresponding fictitious function value u_{-1} and imposing $u_{-1} = u_1$. The discretized equation is then enforced at each of the nodes $x_0, \ldots, x_{N-1}$.
 (a) Express the conditions for the numerical solution in the form of a linear system $\mathbf{A}\mathbf{u} = \mathbf{b}$ where $\mathbf{u} := (u_0, \ldots, u_{N-1})^\top$ and $\mathbf{b} := (h^2 f(x_0), \ldots, h^2 f(x_{N-1}))^\top$; i.e., determine the matrix $\mathbf{A}$.
 (b) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$.
 (c) Write down the Jacobi iteration scheme for the numerical solution $\mathbf{u}$.
 (d) What can you say about the convergence of the Jacobi iteration scheme?

(4) Consider the equation $u_t + (a(x)u)_x = 0$ where $a(x)$ is a given smooth function on $[-\pi, \pi]$, on which the equation is to be solved with periodic boundary conditions. The Lax-Friedrichs scheme for this equation reads (with grid spacing Δx and time step Δt)

$$u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) + \frac{\Delta t}{2\Delta x}(a_{j+1}u_{j+1}^n - a_{j-1}u_{j-1}^n) = 0, \quad a_j := a(j\Delta x).$$

 (a) For the special case that $a(x) = a$, a constant function, show that the scheme is stable in the ℓ_∞-norm provided that $|a|\Delta t \le \Delta x$.

(b) Returning to the general case, show that the scheme is stable in the ℓ_1-norm $|u_0| + \cdots + |u_{N-1}|$ provided that $\Delta t \max_{-\pi \le x \le \pi} |a(x)| \le \Delta x$.

(5) Consider the scheme

$$u_j^{n+1} = u_j^n + a\frac{\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{1}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

for a partial differential equation on the spatial domain $\mathbb{R}$, assuming decaying boundary conditions $u_j^n \to 0$ as $j \to \pm\infty$ for every n.

(a) Using Fourier analysis, calculate the amplification factor and use it to determine the values of the ratio $\lambda := \Delta t/\Delta x$ for which the scheme is stable in ℓ_2.

(b) Assuming that λ satisfies the condition for stability found in part (a), prove that the scheme is ℓ_2-stable using the energy method.

(c) Suppose that $\Delta t = c^{-1}\Delta x$ for some fixed number $c \neq 0$. With which partial differential equation is the scheme consistent as $\Delta x \to 0$?

(d) Suppose instead that $\Delta t = c^{-1}\Delta x^2$. With which partial differential equation is the scheme now consistent as $\Delta x \to 0$?

QUALIFYING REVIEW EXAM: APPLIED ANALYSIS
FIRST PART, MAY 2019

1. Let $C^1([a,b])$ be the space of real-valued continuously differentiable functions on the interval $[a,b]$. If $f \in C^1([a,b])$, show that

$$\|f\| = \left[\int_a^b \left(|f|^2 + |f'|^2\right) dx\right]^{1/2}$$

is a norm. Is $C^1([a,b])$ a Banach space under this norm?

2. Let X be an inner product space and $\{x_i\}_{i=1}^n$ an orthonormal set in X. Show that the function

$$f(c_1, \ldots, c_n) = \left\| x - \sum_{i=1}^n c_i x_i \right\|$$

is minimized by choosing $c_i = \langle x_i, x\rangle$ for all $x \in X$.

3. Let H be a Hilbert space. Show that a necessary and sufficient condition for the orthonormal set $\{x_n\}$ to be a basis for H is that

$$\langle x, y\rangle = \sum_n \langle x, x_n\rangle\langle x_n, y\rangle \ , \quad \text{for all} \quad x, y \in H \ .$$

4. Let x_n and y_n be sequences in a Hilbert space H. (i) If $x_n \to x$ and $y_n \rightharpoonup y$, show that $\langle x_n, y_n\rangle \to \langle x, y\rangle$. (ii) Prove or give a counter-example: if $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$ then $\langle x_n, y_n\rangle \to \langle x, y\rangle$.

5. Let K be a compact subset of $\mathbb{R}^2$ and $C(K)$ the Banach space of real-valued continuous functions on K with the uniform norm. Consider the integral operator $A : C(K) \to C(K)$ defined by $f(x) \mapsto \int_K k(x,y) f(y) dy$. Suppose that for $x \neq y$, the function k obeys the estimate $|k(x,y)| \le M|x-y|^{\alpha-2}$ where $\alpha \in (0,2]$ and M is a constant. Show that A is a compact operator.