QUALIFYING REVIEW EXAM: APPLIED ANALYSIS
MORNING PART, MAY 2018

(1) Consider the boundary-value problem

$$u'' + k^2 u = \sin(x), \qquad u(0) = u(\pi) = 0,$$

where k is an arbitrary real-valued constant.

(a) Find the solution using the method of undetermined coefficients.

(b) For which values of k does the boundary-value problem have (i) a unique solution, (ii) no solution, (iii) more than one solution?

(c) Assuming that k is such that (i) holds, construct the solution using an appropriate Green's function.

(d) Describe what happens to both the Green's function and the corresponding solution $u(x)$ when k approaches the exceptional values where (ii) or (iii) holds.

(2) Let $\{x_n\}$ be a sequence in a Banach space with norm $\|\cdot\|$. Show that if

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\| < \infty,$$

then $\{x_n\}$ converges.

(3) Let H be a real Hilbert space and $A(x,y)$ a symmetric nonnegative bilinear form on H. Show that

$$|A(x,y)| \leq \sqrt{A(x,x)}\sqrt{A(y,y)}, \quad x, y \in H.$$

(4) Let $X = [0,1]$. Consider the operator

$$A : f \mapsto \int_0^x f(y)\mathrm{d}y.$$

(a) Show that $A : L^2(X) \longrightarrow L^2(X)$.

(b) Compute the $L^2(X)$ operator norm $\|A\|$.

(5) Let $f \in C_0^\infty(\mathbb{R}^n)$ be nonnegative with $\int f \mathrm{d}x = 1$. Define $f_\epsilon(x) := \epsilon^{-n} f(x/\epsilon)$ for $\epsilon > 0$. Show that $f_\epsilon \to \delta$ in the sense of distributions in the limit $\epsilon \to 0$.

QUALIFYING REVIEW EXAM: APPLIED ANALYSIS
AFTERNOON PART, MAY 2018

(1) Consider the initial value problem $y' = Ay, y(0) = y_0$, where $A = \begin{pmatrix} -11 & 9 \\ 9 & -11 \end{pmatrix}$.

   (a) Let u_n be the numerical solution given by Euler's method with step size h and $u_0 = y_0$. What condition on h (if any) ensures that u_n remains bounded as $n \to \infty$?

   (b) Consider the initial condition $y(0) = (1, 0)^\top$. Give an explicit expression for the numerical solution u_n. Assume that $t > 0$ is fixed and $t = nh$.

   (c) Find the limit of the numerical solution u_n as $n \to \infty$ under the same assumption that $t > 0$ is fixed and $t = nh$.

(2) Consider the 2-step Adams-Bashforth method $u_{n+1} = u_n + \frac{1}{2}h(3f(u_n) - f(u_{n-1}))$ for the initial value problem $y' = f(y), y(0) = y_0$. Assume that $u_0 = y_0, u_1 = y(h)$.

   (a) The local truncation error has the form $\tau_n = ch^p y_n^{(q)} + O(h^r)$, where c, p, q, r are constants and $y_n = y(nh)$ is the exact solution. Find c, p, q, r.

   (b) Assume that $t > 0$ is fixed and $t = nh$. If the step size h is reduced by a factor of $\frac{1}{2}$, then the error in the numerical solution $|u_n - y_n|$ is reduced by what factor?

   (c) Is the negative real axis contained in the region of absolute stability?

(3) Consider the system of linear equations $2x - y = 1, 2y - x = 1$.

   (a) Starting from initial iterate $(x, y) = (0, 0)$, take one step each of Jacobi's method, the Gauss-Seidel method, and SOR with relaxation parameter $\omega = 3/2$. Which method gives the smallest error after one step?

   (b) Which of these methods (if any) converge as the number of iterations tends to infinity?

(4) Consider the heat equation $v_t = v_{xx}$ with initial condition $v(x, 0) = v_0(x)$ on the real axis. Let u_j^n be the numerical solution given by the finite-difference scheme

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \quad u_j^0 = v_0(jh),$$

where h, k are the space step and time step, respectively, and $u_j^n \approx v(jh, nk)$. Set $\lambda = k/h^2$ and show that the scheme is stable in the ∞-norm *if and only if* $\lambda \leq \frac{1}{2}$.

(5) Consider the initial value problem for the scalar wave equation $v_t + cv_x = 0$ on the real line. Let u_j^n be the numerical solution given by the Lax-Wendroff scheme with space step h and time step k, and assume that h, k are chosen so that $\lambda = k/h$ is fixed.

   (a) Find the amplification factor of the scheme.

   (b) Show that the scheme is stable in the 2-norm under the assumption that $|c|\lambda \leq 1$.