University of Michigan QR Exam in Applied Analysis (556) May 2014

1. (a) (5 pts) Assume f is 2π-periodic with Fourier series $\sum_{-\infty}^{\infty} f_k e^{ikx}$, and $|f_k| \le 2^{-k}$. How many continuous derivatives does f have?

   (b) (5 pts) Give a counterexample to the following statement: If $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$, then $\int_{-\infty}^{\infty} f_n(x)dx \to \int_{-\infty}^{\infty} f(x)dx$.

   (c) (10 pts). Find the x and t dependences of the general term in the Fourier-series solution to $\partial_t u = k\partial_{xx}u$, $k > 0$, on the interval $0 \le x \le 1$, with boundary conditions $\partial_x u(0,t) = \partial_x u(1,t) = 0$.

2. (a) (5 pts) Let a, b be elements of a normed vector space. Show that $|\|a\| - \|b\|| \le \|a - b\|$.

   (b) (10 pts) Let $\{v_1, \dots v_m\}$ be an orthonormal set in $\mathbb{C}^k$.

   Show that for any $a \in \mathbb{C}^k$, there is a unique set of constants $\{c_1, \dots, c_m\}$ such that $a - \sum_1^m c_j v_j$ is orthogonal to all the v_j's, and determine the constants explicitly.

   (c) (5 pts) Find the expansion of the constant function 1 in terms of an orthonormal basis of trigonometric functions on $[0, 2\pi]$.

3. (20 pts) Let $\partial_{xx}u + \partial_{yy}u = 0$ for $x \in \mathbb{R}, y > 0$ with boundary condition $u(x,0) = f(x)$. Show that, under suitable assumptions on f, a solution is:

$$u(x,y) = \int_{-\infty}^{\infty} \frac{yf(x-t)}{\pi(t^2+y^2)} dt.$$

(Hint: consider the Fourier transform). Is this solution unique?

4. (a) (5 pts) Find a sequence of functions which converges weakly to the Dirac delta function on $\mathbb{R}$, and show that it does so.

   (b) (10 pts) In $\mathbb{R}^n$, show that $|\mathbf{x}|^2 \nabla^2 \delta(\mathbf{x}) = 2n\delta(\mathbf{x})$.

   (c) (5 pts) Show that $u(x) = H(x)$, the Heaviside function, is a weak solution of $x\frac{du}{dx} = 0$ for $x \in \mathbb{R}$.

5. (a) (15 pts) Find the Green's function for the ODE $u'' = f(x), 0 < x < 1$ with boundary conditions $B_1 u = u'(0) - u(1) = 0$ and $B_2 u = u'(1) = 0$.

   (b) (5 pts) Give the solution to $\nabla^2 N(x) = \delta(x)$ in $\mathbb{R}^3$ and show that it is the solution.

May 2014

QR Exam in Numerical Analysis

1. Are the following statements true or false? Justify your answer for full credit.

   (a) The Lax-Wendroff scheme is l_∞-stable iff $|\nu| \le 1$.

   (b) The problem $u'' = f(x)$, $u(0) = u(1) = 0$, is solved by centered differencing approximation. Then Gauss-Seidel converges more rapidly as $h \to 0$.

   (c) $u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) + kD_+D_-u_j^n$ approximates the heat equation.

   (d) The iteration $u_j^{(k+1)} = \frac{1}{2}\left(u_{j-1}^{(k)} + u_{j+1}^{(k)}\right) - h^2 f_j$ converges for any initial guess u_j^0.

2. Consider the implicit 2-step method

$$\alpha u^n + \beta u^{n-1} + \gamma u^{n-2} = hf(u^n)$$

   for the problem $u' = f(u)$, $u(0) = u_0$.

   (a) Find α, β and γ which yield a second order scheme.

   (b) Show that the method in part (a) is zero-stable.

   (c) Show that the method in part (a) is A-stable.

3. The linearized model for a swinging pendulum with damping is given by

$$\theta''(t) = -a\theta(t) - b\theta'(t)$$

   where $a > 0$, and $b \ge 0$ is the friction coefficient.

   (a) Define $U_1 = \theta$ and $U_2 = \theta'$ and write the above second order equation as a first order system.

   (b) The system in part (a) is solved by Euler's method. If $a = 4$ and $b = 5$, what restriction on the step size k is needed in order to ensure that the scheme is stable?

   (c) What restriction on k is needed in the case of undamped motion ($b = 0$)? What would be a more suitable scheme for this case?

4. Consider the Crank-Nicolson scheme for the heat equation

$$\frac{u_j^{n+1} - u_j^n}{k} = D_+D_-\left(\frac{u_j^n + u_j^{n+1}}{2}\right).$$

   (a) Compute the LTE.

   (b) Show that if $r = k/h^2 \le 1$, $||u^{n+1}||_\infty \le ||u^n||_\infty$.

   (c) Use the energy method to show that the scheme is stable in the l_2-norm. Under what conditions on $r = k/h^2$?

5. The box scheme for $u_t + au_x = 0$ is defined by

$$\frac{(u_j^{n+1} - u_j^n) + (u_{j+1}^{n+1} - u_{j+1}^n)}{2\Delta t} + a\frac{(u_{j+1}^{n+1} - u_j^{n+1}) + (u_{j+1}^n - u_j^n)}{2\Delta x} = 0$$

Use the Fourier method to analyze the stability of this scheme in the l_2 norm.