UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS
Qualifying Review Examination in Applied Analysis
4 January 2005: Morning Session, 9:00-12:00

1. Suppose that $f(x)$ has two continuous derivatives on $[0, L]$, and that $f(0) = f(L) = 0$.
   (a) Prove that for the coefficients b_n in the Fourier sine series of $f(x)$, the estimate
   $$|b_n| \leq \frac{C}{n^2}$$
   holds for some $C > 0$ depending on f but not on n. Give an upper bound for the constant C in terms of the L^2 norm of f''.
   (b) Prove that the mean square (that is, the L^2 norm squared) error in approximating f by the sum of the first N terms in its Fourier sine series is bounded by K/N^3 for some constant $K > 0$ depending on f but not on N. Give an upper bound for the constant K in terms of the constant C from part (a).

2. Prove the distributional identity
   $$\Delta \frac{1}{r} = -4\pi\delta(\mathbf{x})$$
   where $\mathbf{x} = (x, y, z)$, $r = |\mathbf{x}|$, and Δ is the Laplace operator in $\mathbb{R}^3$.

3. Let
   $$\mathbf{A} = \frac{1}{2}\begin{pmatrix} 1+e & e-1 \\ e-1 & 1+e \end{pmatrix}.$$
   (a) Find $\ln(\mathbf{A})$.
   (b) Show that there is some constant C such that
   $$C\int_0^\infty \frac{t^{-1/2}\,dt}{t+\mathbf{A}} = \mathbf{A}^{-1/2}.$$
   Find the value of the constant C. Will C change if $\mathbf{A}$ is a different matrix? Explain.

4. Consider the initial/boundary value problem for the heat equation:
   $$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}\,, 0 < x < L\,, t > 0\,,$$
   with boundary conditions $T(0, t) = T(L, t) = 0$ and initial condition $T(x, 0) \equiv 1$.
   (a) Solve this problem by separation of variables.
   (b) Give some interpretation of the fact that the given initial data does not satisfy the boundary conditions. Is this a problem? Why or why not? What happens to the solution from part (a) as $t \downarrow 0$?

5. For the heat equation initial/boundary value problem from Problem 4, with more general initial data $T(x, 0) = T_0(x)$, give a precise formulation of stability, the notion that "small changes in initial data $T_0(x)$ lead to small changes in the solution." Outline a proof of the statement you formulate.

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4 January 2005: Afternoon Session, 2:00-5:00

1. Let $f(x)$ be a twice continuously differentiable function on $[a,b]$. Find and prove an upper bound on the error in the numerical approximation

$$\int_a^b f(x)\,dx \approx \frac{b-a}{N}\sum_{k=1}^{N} f\left(a + \left(\frac{b-a}{N}\right)(k-1/2)\right),$$

and thus determine the rate that the error tends to zero as N increases.

2. Consider the two-step predictor method given by

$$y^{n+2} + a_1 y^{n+1} + a_0 y^n = h\left[b_0 f(y^n) + b_1 f(y^{n+1})\right]$$

for solving numerically the initial-value problem

$$\frac{dy}{dt} = f(y)\,,\, y(0) = y^0\,.$$

   (a) Determine the constants a_0, b_0, and b_1 in terms of the remaining constant a_1 such that the method has order at least 2.

   (b) For which values of a_1 is the method found in part (a) stable for $f(y) \equiv 0$?

3. For numerical solution of the autonomous equation $dy/dt = f(y)$, consider the following implicit scheme

$$3y^{n+1} - 4y^n + y^{n-1} = 2\Delta t f(y^{n+1})\,.$$

   (a) Show that the region of absolute stability for this scheme contains the negative real axis.

   (b) Use the Gerschgorin Disk Theorem to estimate the eigenvalues of the banded matrix

$$\underline{\underline{A}} = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

   (c) The heat equation $u_t = u_{xx}$ is to be solved with zero boundary conditions by means of a discretization $u_j^n \approx u(j\Delta x, n\Delta t)$ and the scheme

$$\frac{3u_j^{n+1} - 4u_j^n + u_j^{n-1}}{2\Delta t} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2}\,.$$

   Determine conditions on Δt and Δx under which this scheme is stable.

4. Consider the explicit scheme

$$r_j^{n+1} = \frac{1}{2}\left(r_{j+1}^n + r_{j-1}^n\right) + \frac{\Delta t}{2\Delta x}\left(s_{j+1}^n - s_{j-1}^n\right), \qquad \text{and} \qquad s_j^{n+1} = \frac{1}{2}\left(s_{j+1}^n + s_{j-1}^n\right) + \frac{\Delta t}{2\Delta x}\left(r_{j+1}^n - r_{j-1}^n\right),$$

where periodic boundary conditions are imposed on the sequences $\{r_j^n\}$ and $\{s_j^n\}$ regarding their dependence on j for each fixed n.

(a) With which system of partial differential equations is this scheme consistent?

(b) What is the order of the local truncation error for this scheme?

(c) Find conditions on Δt and Δx sufficient for this scheme to satisfy the von Neumann stability condition.

(d) Find conditions on Δt and Δx under which this scheme is L^2-stable.

5. Consider solving the heat equation $u_t = u_{xx}$ with boundary conditions $u(0,t) = u(1,t) = 0$ and the numerical scheme

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x^2}\left(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}\right), \qquad u_0^n = u_N^n = 0\,.$$

(a) This is an implicit scheme. Hence, to advance the solution from time $t = n\Delta t$ to $t = (n+1)\Delta t$, consider the iteration

$$u_j^{n+1,k+1} = u_j^n + \frac{\Delta t}{\Delta x^2}\left(u_{j-1}^{n+1,k} - 2u_j^{n+1,k} + u_{j+1}^{n+1,k}\right),$$

for $k = 0, 1, 2, 3, \ldots$, and with $u_j^{n+1,0}$ arbitrary. The idea is that by repeating this iteration, one should have $u_j^{n+1} = u_j^{n+1,\infty}$. Under what conditions on Δt and Δx does convergence occur as $k \to \infty$? Explain.

(b) Use the energy method to show that this scheme is stable in L^2 under certain conditions on Δt and Δx. What are these conditions?