Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
May 09, 2024; Morning Session

Note: all L^p spaces appearing below are with respect to the Lebesque measure.

Problem 1. Construct a function $f \in L^2(\mathbb{R})$ such that $f \notin L^p([a,b])$ for each $-\infty < a < b < +\infty$ and each $p > 2$.

Solution. Let $r \in \mathbb{Q}$ and $n \in \mathbb{N}$. Consider a non-negative function

$$f_{r,n}(x) := |x-r|^{-\frac{1}{2}+\frac{1}{n+2}} \mathbb{1}_{[r-1,r+1]}(x)$$

and note that $f_{r,n} \in L^2(\mathbb{R})$ but $f_{r,n} \notin L^p([a,b])$ if $a < r < b$ and $p \geq 2 + \frac{4}{n}$. The set of all pairs (r,n) is countable, let (r_k, n_k), $k \in \mathbb{N}$, be its enumeration. The function

$$f(x) := \sum_{k=1}^{\infty} 2^{-k} \cdot \frac{f_{r_k,n_k}(x)}{\|f_{r_k,n_k}\|_{L_2(\mathbb{R})}}$$

belongs to $L^2(\mathbb{R})$ (we use the completeness of this space here) but does not belong to $L^p([a,b])$ if $a < r_k < b$ and $p \geq 2 + \frac{4}{n_k}$ (since the corresponding integral diverges). The latter implies that $f \notin L^p([a,b])$ for all $a < b$ and $p > 2$ as desired.

Remark. In fact, there is no need to include the additional parameter n into the construction. Instead, one can start with the functions

$$f_r(x) := |x-r|^{-\frac{1}{2}} |\log|x-r||^{-1} \cdot \mathbb{1}_{[r-1,r+1]}(x)$$

that belong to $L^2(\mathbb{R})$ but not to $L^p([a,b])$ with $a < r < b$ and $p > 2$.

Problem 2. Let $E \subset \mathbb{R}$ be a measurable set such that $\lambda_1(E) > 0$, where λ_1 stands for the Lebesgue measure on $\mathbb{R}$.

(a) Prove that there exists an interval $\emptyset \neq I \subset \mathbb{R}$ such that $\lambda_1(E \cap I) \geq \frac{3}{4}\lambda_1(I)$.

(b) Prove that 0 is an interior point of the set $E - E := \{x - y \,|\, x, y \in E\}$.

Solution. **(a)** The countable additivity implies $\lambda_1(E \cap (n, n+1)) > 0$ for some $n \in \mathbb{Z}$. Without loss of generality, below we assume that $n = 0$ and $E = E \cap (0,1)$. There is nothing to prove if $\lambda_1(E) \geq \frac{3}{4}$, thus we can also assume that $\lambda_1(E) < \frac{3}{4}$. In this case one can use the regularity of the Lebesgue measure in order to find an *open* set U such that $E \subset U \subset (0,1)$ and $\lambda_1(U) < \frac{4}{3}\lambda_1(E)$. Each open subset of the real line is an at most countable union of disjoint open intervals: $U = \bigsqcup_k I_k$. If we had $\lambda_1(E \cap I_k) < \frac{3}{4}\lambda_1(I_k)$ for all k, then the countable additivity would imply that

$$\lambda_1(E) = \lambda_1(E \cap U) = \textstyle\sum_k \lambda_1(E \cap I_k) < \frac{3}{4} \sum_k \lambda_1(I_k) = \frac{3}{4}\lambda_1(U) < \lambda_1(E),$$

which is a contradiction.

(b) We know from part (a) that there exists an interval I such that $\lambda_1(E \cap I) \geq \frac{3}{4}\lambda_1(I)$. Without loss of generality (shift and rescale the set E, which results in a rescaling of the difference set $E - E$) let us assume that $I = (0,1)$. Note that $0 \in E - E$ and assume, that $t \notin E - E$ for some $t \in (0, \frac{1}{2})$; the case $t \in (-\frac{1}{2}, 0)$ is similar. This assumption means that the sets E and $E + t$ are disjoint and hence

$$\tfrac{3}{2} \leq 2\lambda_1(E) = \lambda_1(E) + \lambda_1(E+t) = \lambda_1(E \cup (E+t)) \leq \lambda_1((0, 1+t)) = 1 + t,$$

a contradiction. Therefore, $(-\frac{1}{2}, \frac{1}{2}) \subset E - E$ and 0 is an interior point of $E - E$.

Problem 3. Let $f \in L^1([0,1])$ and let $g : [0,1] \to \mathbb{R}$ be a bounded increasing function on $[0,1]$ such that for all $0 \le a < b \le 1$ one has

$$\left| \int_a^b f(x)dx \right|^2 \le (g(b) - g(a))(b-a)\,.$$

Prove that $f \in L^2([0,1])$.

Solution. Let $b = a + h$ and rewrite the assertion as

$$\left| \frac{1}{h} \int_a^{a+h} f(x)dx \right|^2 \le \frac{g(a+h) - g(a)}{h}\,.$$

We now use the following two facts:

- monotone functions are almost everywhere differentiable;
- if $f \in L^1$, then $\frac{1}{h}\int_a^{a+h} f(x)dx \to f(a)$ as $h \downarrow 0$ for almost all a.

Sending $h \downarrow 0$ one easily concludes that $|f(a)|^2 \le g'(a)$ almost everywhere on $[0,1]$. Note that $g' \in L^1([0,1])$ since $\int_0^1 g'(x)dx \le g(1) - g(0)$. Therefore, $f \in L^2([0,1])$.

Problem 4. Let a sequence of functions $f_n \in L^3([-1,1])$ be such that $f_n \underset{n\to\infty}{\to} f$ a.e. on $[-1,1]$ and $\|f_n\|_3 \le C < +\infty$. Prove that

$$\iint_{[-1,1]^2} \frac{f_n(x)f_n(y)}{(x^2+y^2)^{1/2}}\,dxdy \underset{n\to\infty}{\to} \iint_{[-1,1]^2} \frac{f(x)f(y)}{(x^2+y^2)^{1/2}}\,dxdy\,.$$

Solution. First, note that $f \in L^3([-1,1])$ due to Fatou's lemma:

$$\int_{-1}^1 |f(x)|^3 dx \le \liminf_{n\to\infty} \int_{-1}^1 |f_n(x)|^3 dx \le C^3\,.$$

Second, a standard argument implies the convergence $f_n \to f$ in L^p for all $p < 3$. To prove this, one writes

$$\int_{-1}^1 |f_n(x) - f(x)|^p dx = \int_{\{|f_n - f| \le M\}} |f_n(x) - f(x)|^p dx + \int_{\{|f_n - f| \ge M\}} |f_n(x) - f(x)|^p dx$$

and applies the Hölder inequality to the second term:

$$\begin{aligned}
&\int_{\{|f_n-f|\ge M\}} |f_n(x) - f(x)|^p dx \\
&\qquad \le \left(\int_{\{|f_n-f|\ge M\}} |f_n(x) - f(x)|^3 dx \right)^{\frac{p}{3}} \cdot \left(\int_{\{|f_n-f|\ge M\}} dx \right)^{1-\frac{p}{3}} \\
&\qquad \le \|f_n - f\|_3^p \cdot (\lambda_1(\{x : |f_n(x) - f(x)| \ge M\}))^{1-\frac{p}{3}} \\
&\qquad \le \|f_n - f\|_3^p \cdot \left(\frac{\|f_n - f\|_3}{M} \right)^{3-p} = \frac{\|f_n - f\|_3^3}{M^{3-p}}\,.
\end{aligned}$$

Therefore,

$$\int_{-1}^1 |f_n(x) - f(x)|^p dx \le \int_{\{|f_n - f| \le M\}} |f_n(x) - f(x)|^p dx + \frac{(2C)^3}{M^{3-p}}$$

and one can first choose M big enough and then n big enough in order to make this sum arbitrarily small. (Note that the integrand in the first term is uniformly bounded and hence the dominated convergence theorem applies.)

It is also easy to see that the convergence $f_n \to f$ in $L^p([-1,1])$ implies the convergence of the functions $f_n(x)f_n(y)$ of two variables to $f(x)f(y)$ in $L^p([-1,1]^2)$. In particular, this convergence holds, e.g., for $p = \frac{5}{2}$. Due to the duality of L^p and $L^{p/(p-1)}$ (or simply to the Hölder inequality), it remains to show that the function $g(x,y) := (x^2+y^2)^{-1/2}$ belongs to the space $L^{5/3}([-1,1]^2)$. This is straightforward in the polar coordinates:

$$\int_{[-1,1]^2} |g(x,y)|^{\frac{5}{3}} dxdy \;\le\; 2\pi \int_0^{\sqrt{2}} \frac{rdr}{r^{5/3}} < +\infty\,.$$

Problem 5. Let $f, g : [0,1] \to [0,1]$ be absolutely continuous functions.

(a) Is it always true that the composition $h := f \circ g$ is absolutely continuous? (Give a proof or provide a justified counterexample.)

(b) Assume, in addition, that $h = f \circ g$ is of bounded variation. Prove that in this case h is absolutely continuous. [*Hint:* you may start by proving that the function h sends sets of zero measure to sets of zero measure. For the remaining (harder!) part of the argument you can rely (without justification) upon the fact that $\int_0^1 \#\{x : h(x)=y\}\, \lambda_1(dy)$ equals the total variation of h.]

Solution. **(a)** No, the composition of absolutely continuous functions is *not* necessarily absolutely continuous. Moreover, it is not necessarily of bounded variation, an example is given by $f(x) = \sqrt{x}$ and $g(x) = x^2|\cos\frac{1}{x}|$. (Both functions $f(x) = \sqrt{x}$ and $\widetilde{g}(x) := x^2\cos\frac{1}{x}$ are differentiable on $(0,1]$ with L^1 derivatives, thus absolutely continuous. Therefore, $g(x) = |\widetilde{g}(x)|$ is also absolutely continuous.) Indeed, the composition $h = f \circ g$ is given by

$$h(x) = x|\cos\tfrac{1}{x}|^{1/2}, \quad x \in [0,1];$$

in particular $h(\frac{1}{\pi n}) = \frac{1}{\pi n}$ and $h(\frac{1}{\pi(n+\frac{1}{2})}) = 0$ for all $n \in \mathbb{N}$, which means that the total variation of h on $[0,1]$ is bounded from below by $\sum_{n=1}^{\infty} \frac{1}{\pi n} = +\infty$.

(b) As suggested in the hint, we begin with proving that $h = f \circ g$ sends sets of zero measure to sets of zero measure. It is enough to show that the same holds for both f and g or, more generally, to each absolutely continuous function $f : [0,1] \to [0,1]$.

Let $E \subset [0,1]$ is such that $\lambda_1(E) = 0$; in order to prove that $\lambda_1(f(E)) = 0$ we can also assume that $E \subset (0,1)$. As f is absolutely continuous, for each $\varepsilon > 0$ there exists $\delta > 0$ such that the assertion $\sum_k |t_k - s_k| < \delta$ implies $\sum_k |f(t_k) - f(s_k)| \le \varepsilon$ provided that the intervals $(\min\{s_k, t_k\}, \max\{s_k, t_k|)$ are disjoint; note that the summation over countably many intervals is allowed (e.g., by taking the limit of finite sums).

For each $\delta > 0$ there is an open set $U \supset E$ such that $\lambda_1(E) < \delta$. Open subsets of the real line are at most countable disjoint unions of intervals, so $U = \bigsqcup_k (a_k, b_k)$. Let $s_k := \min_{[a_k,b_k]} f$ and $t_k := \max_{[a_k,b_k]} f$. Then, $\sum_k |t_k - s_k| \le \sum_k (b_k - a_k) < \delta$ and

$$\lambda_1(f(E)) \;\le\; \lambda_1(f(U)) \;\le\; \textstyle\sum_k \lambda_1(f(a_k,b_k)) \;\le\; \sum_k |f(t_k) - f(s_k)| \;\le\; \varepsilon\,.$$

As ε can be taken arbitrary small, this proves that $\lambda_1(f(E)) = 0$ if $\lambda_1(E) = 0$. The same argument applies to g, which implies that the composition $h = f \circ g$ also sends sets of zero measure to sets of zero measure.

The remaining part of the proof is considerably harder (see the next page). Providing the full solution to this question was not required for the 100/100 score.

Assume (proof by contradiction) that $h = f \circ g$ is *not* absolutely continuous. Then, there exists $\varepsilon_0 > 0$ and a sequence of collections of disjoint intervals $\{(a_{k,n}, b_{k,n})\}_{k=1,\dots,K_n}$, $n \in \mathbb{N}$, such that

$$\sum_{k=1}^{K_n} (b_{k,n} - a_{k,n}) < 2^{-n} \quad \text{and} \quad \sum_{k=1}^{K_n} \left(\sup_{(a_{k,n},b_{k,n})} h - \inf_{(a_{k,n},b_{k,n})} h \right) \geq \varepsilon_0.$$

Denote

$$E := \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k=1}^{K_n} (a_{k,n}, b_{k,n}),$$

it is clear that $\lambda_1(E) = 0$ (note that $\lambda_1(E) \leq \sum_{n=m}^{\infty} 2^{-n}$ for all $m \in \mathbb{N}$).

Note that we have the following inequality:

$$\phi_n(y) := \textstyle\sum_{k=1}^{K_n} \mathbb{1}_{h((a_{k,n},b_{k,n}))}(y) \ \leq \ \phi(y) := \#\{x \in [0,1] : h(x) = y\}\,.$$

Assume that $y \in [0,1]$ is such that $\phi(y) < +\infty$. In this case it is easy to see that $\phi_n(y) \to 0$ unless $y \in h(E)$: indeed, for such y the equation $\phi(x) = y$ has only finitely many solutions and hence there is $x \in f^{-1}(y)$ such that $x \in \bigcup_{k=1}^{K_n} (a_{k,n}, b_{k,n})$ holds for infinitely many indices n, which implies $x \in E$ and $y \in f(E)$.

The desired contradiction now comes from the following facts with the help of the dominated convergence theorem:

- by our assumption we have $\int_0^1 \phi_n(y) dy \geq \varepsilon_0 > 0$ for all n;
- the majorant ϕ is summable (since h is of bounded variation);
- $\phi_n(y) \to 0$ as $n \to \infty$ almost everywhere: indeed, $\lambda_1(\{y : \phi(y) = +\infty\}) = 0$ since $\phi \in L^1([0,1])$, and $\lambda_1(h(E)) = 0$ since $\lambda_1(E) = 0$ (as discussed above).

The proof is complete.

Remark. The function $\#\{x : f(x) = y\}$ is known as the *Banach indicatrix* of f.