Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
May 09, 2024; Morning Session

Note: all L^p spaces appearing below are with respect to the Lebesque measure.

Problem 1. Construct a function $f \in L^2(\mathbb{R})$ such that $f \notin L^p([a,b])$ for each $-\infty < a < b < +\infty$ and each $p > 2$.

Problem 2. Let $E \subset \mathbb{R}$ be a measurable set such that $\lambda_1(E) > 0$, where λ_1 stands for the Lebesgue measure on $\mathbb{R}$.
(a) Prove that there exists an interval $\emptyset \neq I \subset \mathbb{R}$ such that $\lambda_1(E \cap I) \geq \frac{3}{4}\lambda_1(I)$.
(b) Prove that 0 is an interior point of the set $E - E := \{x - y \,|\, x, y \in E\}$.

Problem 3. Let $f \in L^1([0,1])$ and let $g : [0,1] \to \mathbb{R}$ be a bounded increasing function on $[0,1]$ such that for all $0 \leq a < b \leq 1$ one has

$$\left| \int_a^b f(x)dx \right|^2 \leq (g(b) - g(a))(b - a)\,.$$

Prove that $f \in L^2([0,1])$.

Problem 4. Let a sequence of functions $f_n \in L^3([-1,1])$ be such that $f_n \underset{n\to\infty}{\to} f$ a.e. on $[-1,1]$ and $\|f_n\|_3 \leq C < +\infty$. Prove that

$$\iint_{[-1,1]^2} \frac{f_n(x)f_n(y)}{(x^2+y^2)^{1/2}}\,dxdy \underset{n\to\infty}{\to} \iint_{[-1,1]^2} \frac{f(x)f(y)}{(x^2+y^2)^{1/2}}\,dxdy\,.$$

Problem 5. Let $f, g : [0,1] \to [0,1]$ be absolutely continuous functions.
(a) Is it always true that the composition $h := f \circ g$ is absolutely continuous? (Give a proof or provide a justified counterexample.)
(b) Assume, in addition, that $h = f \circ g$ is of bounded variation. Prove that in this case h is absolutely continuous. [*Hint:* you may start by proving that the function h sends sets of zero measure to sets of zero measure. For the remaining (harder!) part of the argument you can rely (without justification) upon the fact that $\int_0^1 \#\{x : h(x) = y\}\, \lambda_1(dy)$ equals the total variation of h.]