Qualifying Review Exam
Complex Analysis
January 2024

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

(1) Find all solutions of $\cos z = 1 + 100z^2$ in the unit disk $|z| < 1$.

(2) Find

$$\sup \left\{ |f(1)| : f \text{ is holomorphic on } \mathbb{C} \setminus \{0\} \text{ and satisfies } |f(z)| \le 7|z|^{-3/2} \right\}.$$

(3) Let $f_k : \mathbb{D} \to \mathbb{C}$ be a sequence of holomorphic functions forming a normal family (that is to say, every subsequence of (f_k) has a further subsequence converging uniformly on each compact subset of $\mathbb{D}$). Further, let $h_k : \mathbb{D} \to \mathbb{D}$ be holomorphic functions satisfying $h_k(0) = 0$. Prove that the functions

$$g_k(z) = f_k\left(h_k(z)\right)$$

form a normal family.

(4) Let $D_1, D_2 \subset \mathbb{C}$ be disks with the property that the circles Bd D_1, Bd D_2 intersect in exactly two points. Under what additional hypothesis will there exist a bijective *rational* map from $D_1 \cap D_2$ to $\mathbb{D}$?

(5) Suppose that f is holomorphic on $\{z \in \mathbb{C} : |z| > r\}$ for some $r < 1$. Suppose further that $zf(z) \to 1$ as $z \to \infty$.

(a) Evaluate $\displaystyle\int_{|z|=1} z f'(z)\, dz$.

(b) Show that $\displaystyle\int_{|z|=1} |f'(z)||dz| \ge 2\pi$.

(c) When does equality hold in (b)?