Complex Analysis Qualifying Review

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1. Let $\alpha > 0$ be a real number.

(a) Prove that if $\alpha \leq 1$, then there exists an analytic function f on the unit disc such that $f(\frac{1}{n}) = \frac{1}{n+\alpha}$ for all integers $n \geq 1$.
(b) Prove that if $\alpha > 1$ and f is an analytic function on the unit disc, then there exist only finitely many integers $n \geq 1$ such that $f(\frac{1}{n}) = \frac{1}{n+\alpha}$.

(a) Simply take $f(z) = \frac{z}{1+\alpha z}$. This is analytic on the unit disc since $1 + \alpha z \neq 0$ there, and it has the required properties.
(b) The function $g(z) = \frac{z}{1+\alpha z}$ is analytic near the origin and satisfies $g(\frac{1}{n}) = \frac{1}{n+\alpha}$ for all large enough integers n. Now suppose there exists an analytic function f on the unit disc such that $f(\frac{1}{n}) = \frac{1}{n+\alpha}$ for infinitely many integers $n \geq 1$. The function $h = f - g$ is then analytic near the origin and satisfies $h(\frac{1}{n}) = 0$ for infinitely many n. Thus $h = 0$ near the origin, or else the zeros of h would be isolated. By uniqueness of analytic continuation, we get $f(z) = g(z)$, but this is a contradiction since g has a pole at $z = -1/\alpha$, which lies in the unit disc.

2. Does there exist an entire function f (i.e. f is analytic in the whole complex plane) such that the inequality

$$\tfrac{1}{2}|z|^{3/2} - |z| \leq |f(z)| \leq 2|z|^{3/2} + \tfrac{7}{2}|z|$$

holds for all z outside a compact set? Justify your answer.

The answer is no, as can be seen using the following argument. Suppose f exists, and pick $R > 0$ such that the inequality holds for $|z| \geq R$. Pick any $z \in \mathbb{C}$ and pick $r > R + |z|$. The Cauchy estimates give

$$|f''(z)| \leq \frac{2!}{r^2} \max_{|w-z|=r} |f(w)| \leq \frac{4(r+|z|)^{3/2} + 7(r+|z|)}{r^2},$$

which tends to 0 as $r \to \infty$. Thus $f(z) = az + b$ for some constants a, b. But then $|f(z)| \leq |az+b| < \frac{1}{2}|z|^{3/2} - |z|$ if $|z|$ is large enough, a contradiction.

3. Find all analytic functions f on the unit disc $\mathbb{D}$ such that $f(0) = 1$, $f(\frac{1}{2}) = 3$, and $\mathrm{Re} f(z) > 0$ for all $z \in \mathbb{D}$.

The Möbius transformation $z \mapsto \frac{1-z}{1+z}$ takes the right half plane to the unit disc. Thus the analytic function $g(z) := \frac{1+f(z)}{1-f(z)}$ sends the unit disc to itself. We have

$g(0) = \frac{1-1}{1+1} = 0$, so by the Schwarz lemma, $|g(z)| \le |z|$ for all $z \in \mathbb{D}$. Moreover, $g(\frac{1}{2}) = \frac{1-3}{1+3} = -\frac{1}{2}$, so if $z_0 = \frac{1}{2}$, then $|g(z_0)| = |z_0|$. The Schwarz lemma then also gives $g(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Since $g(\frac{1}{2}) = -\frac{1}{2}$ we have $\lambda = -1$. Thus $g(z) = -z$, that is, $\frac{1+f(z)}{1-f(z)} = -z$, which amounts to $f(z) = \frac{1+z}{1-z}$.

4. Use complex integration to compute the real integral $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$.

We compute a complex integral over the unit circle $|z| = 1$, using the parametrization $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1})$. Thus

$$I := \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_{|z|=1} \frac{\frac{dz}{iz}}{2 + \frac{1}{2}(z + z^{-1})} = \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2 + 4z + 1}.$$

Here the integrand has simple poles at $-2 \pm \sqrt{3}$, and no other poles. The pole $z_+ = -2 + \sqrt{3}$ satisfies $|z_+| < 1$ whereas the other one, $z_- = -2 - \sqrt{3}$ satisfies $|z_-| > 1$. The residue of the rational function $\frac{1}{z^2+4z+1}$ at z_+ is given by $\frac{1}{2z_+ + 4} = \frac{1}{2\sqrt{3}}$. By the residue theorem, the requested integral is equal to

$$I = \frac{2}{i} 2\pi i \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$

5. Let D be the (open) square with corners at $\pm 1 \pm i$. Find the number of solutions to the equation $e^z = 3z^{2020}$ in D, counted with multiplicity.

We apply Rouché's theorem to $f(z) = 3z^{2020}$ and $g(z) = -e^z$. On the boundary ∂D of the square, we have $|z| \ge 1$, and hence $|f(z)| \ge 3$, whereas $\mathrm{Re} z \le 1$, and hence $|g(z)| \le e^1 = e$. Thus $|f(z)| > |g(z)|$ on ∂D, so by Rouché's theorem, f and $f + g$ have the same number of zeros, taken with multiplicity in D. Since f has 2020 zeros, so has $f + g$, which means that the equation $e^z = 3z^{2020}$ has 2020 solutions with multiplicity.