Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
May 4, 2023, 9.00 am-12.00 pm

Problem 1: Let μ be a finite Borel measure on $[0,1]$ and $f:[0,1]\to[0,\infty)$ an integrable function with respect to μ. Suppose further that

$$\int_A |f|d\mu \ \le\ \sqrt{\mu(A)} \quad \text{for all Borel sets } A\subset[0,1]\ .$$

Prove that $|f|^p$ is integrable with respect to μ provided $1\le p<2$.

Problem 2: Let $f:(0,1)\to\mathbb{R}$ be a Lebesgue measurable function which satisfies the inequality $\int_0^1 t^3 f(t)^4\ dt<\infty$. Prove that

$$\lim_{x\to 0}\frac{1}{|\log x|^{3/4}}\int_x^1 f(t)\ dt \ =\ 0\ .$$

Problem 3: Suppose A is a Lebesgue measurable subset of $\mathbb{R}$ with positive measure $m(A)>0$. Show that for any b with $0<b<m(A)$ there exists a compact subset $K\subset A$ with $m(K)=b$.

Problem 4: Suppose $f:\mathbb{R}\to\mathbb{R}$ is a continuous function and k an integer such that for all $y\in\mathbb{R}$ the number of distinct solutions to the equation $f(x)=y$ is bounded by k. Prove that the derivative $f'(x)$ exists for a.e. $x\in\mathbb{R}$.

Problem 5: Let f be in $L^1(\mathbb{R})$ and denote by Mf the restricted maximal function

$$Mf(x)\ =\ \max_{0<t<1}\frac{1}{2t}\int_{x-t}^{x+t}|f(x')|\ dx'\ ,\quad x\in\mathbb{R}\ .$$

Prove that

$$M(f*g)(x)\ \le Mf*Mg(x)\ ,\quad x\in\mathbb{R},\ f,g\in L^1(\mathbb{R}),$$

where the operation $*$ denotes convolution:

$$f*g(x)\ =\int_{\mathbb{R}} f(x-y)g(y)\ dy\ ,\quad x\in\mathbb{R}.$$