Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
January 7, 2023, 2.00 pm-5.00 pm

Problem 1: Let $E_k,\ k = 1, 2, \ldots, n$, be measurable subsets of $[0, 1]$ such that each point $x \in [0, 1]$ is contained in at least 5 of the sets $E_k,\ k = 1, \ldots, n$. Prove there exists k such that $m(E_k) \geq 5/n$.

Solution: Letting χ_E be the characteristic function of the set E, we have that $\sum_{k=1}^n \chi_{E_k}(x) \geq 5,\ x \in [0, 1]$. Integrating this inequality we have then that $\sum_{k=1}^n m(E_k) \geq 5$, whence there exists k such that $m(E_k) \geq 5/n$.

Problem 2: For $f \in L^1(\mathbb{R})$ define a sequence of functions $g_n : [0, 1] \to \mathbb{R},\ n = 1, 2, \ldots,$ by

$$g_n(x) \ = \ \sum_{k=1}^n \frac{1}{\sqrt{k}} f(x + \sqrt{k}) \ , \quad n = 1, 2, \ldots$$

Prove that the sequence $g_n,\ n = 1, 2, \ldots,$ is convergent in $L^1([0, 1])$.

Solution: It is sufficient to show that

$$\lim_{m\to\infty} \sum_{k=m^2}^\infty \frac{1}{\sqrt{k}} \int_0^1 |f(x + \sqrt{k})| \ dx \ = \ 0 \quad \text{for integer } m.$$

Note that

$$\int_0^1 |f(x + \sqrt{k})| \ dx \ \leq \ \int_m^{m+2} |f(y)| \ dy \quad \text{if } m^2 \leq k < (m + 1)^2 \ .$$

Hence

$$\sum_{k=m^2}^\infty \frac{1}{\sqrt{k}} \int_0^1 |f(x + \sqrt{k})| \ dx \ \leq \ \sum_{r=m}^\infty \frac{2r + 1}{r} \int_r^{r+2} |f(y)| \ dy \ \leq \ 6 \int_m^\infty |f(y)| \ dy \ .$$

Now use the fact that $f \in L^1(\mathbb{R})$.

Problem 3: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is absolutely continuous on finite intervals, and $g : [0, 1] \to \mathbb{R}$ an integrable function. Define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \int_0^1 f(x - y)g(y) \ dy$. Show that h is also absolutely continuous on finite intervals.

Solution: We need to show that for any finite interval $I = [a, b]$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all sets of disjoint intervals $\{(x_i, x_i') : i = 1, , \dots, n\}$ contained in I one has

$$\sum_{i=1}^{n} |h(x_i') - h(x_i)| \ < \ \varepsilon \quad \text{if} \ \sum_{i=1}^{n} [x_i' - x_i] \ < \ \delta \ .$$

To see this we use the fact that f is ac on the interval $[a - 1, b]$, whence for $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(x_i' - y) - f(x_i - y)| \ < \ \frac{\varepsilon}{\|g\|_{L^1}} \quad \text{if } 0 < y < 1, \ \sum_{i=1}^{n} [x_i' - x_i] \ < \ \delta \ .$$

The result follows on multiplication by $|g(y)|$ and integrating with respect to $y \in [0, 1]$.

Problem 4: Let $f_n, \ n = 1, 2, \dots,$ be a sequence of measurable functions on $[0, 1]$ such that $f_n \to 0$ a.e. and $f_n \in L^3([0, 1]), \ n = 1, 2, \dots,$ with $\sup_{n \geq 1} \|f_n\|_3 < \infty$. Prove there exists p with $1 < p < \infty$ such that

$$(*) \quad \lim_{n \to \infty} \int_0^1 f_n(x) g(x) \ dx \ = \ 0 \quad \text{for all } g \in L^p([0, 1]) \ .$$

Solution: For any $A > 0$ define $f_{n,A}(x) = H(A - |f_n(x)|) f_n(x)$, where $H : \mathbb{R} \to \{0, 1\}$ is the Heaviside function $H(z) = 0$ if $z < 0$, and $H(z) = 1$ if $z > 0$. The dominated convergence theorem then implies that

$$\lim_{n \to \infty} \int_0^1 f_{n,A}(x) g(x) \ dx \ = \ 0 \quad \text{for all bounded } g : [0, 1] \to \mathbb{R} \ .$$

Letting $M = \sup_{n \geq 1} \|f_n\|_3$, we have from the Chebyshev inequality that

$$m(|f_n| > \lambda) \ \leq \ \frac{\|f_n\|_3^3}{\lambda^3} \ \leq \ \frac{M^3}{\lambda^3} \ , \quad \lambda > 0,$$

whence it follows that

$$\int_{|f_n| > A} |f_n(x)| \ dx \ = \ \int_A^\infty m(|f_n| > \lambda) \ d\lambda \ \leq \ \frac{M^3}{2A^2} \ .$$

It follows that $(*)$ holds for all bounded g. We can extend this to $g \in L^p([0, 1])$ with $p = 3/2$ by using the Hölder inequality. Thus for $g \in L^p([0, 1])$ and $\varepsilon > 0$ there exists bounded g_ε such that $\|g - g_\varepsilon\|_p < \varepsilon$. Then

$$\left| \int_0^1 f_n(x) [g(x) - g_\varepsilon(x)] \ dx \right| \ \leq \ \|f_n\|_3 \|g - g_\varepsilon\|_{3/2} \ \leq \ M\varepsilon \ .$$

Problem 5: Let $f(\cdot)$ be an integrable function on $\mathbb{R}^n$ and Mf the corresponding Hardy-Littlewood maximal function

$$Mf(x) \;=\; \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|\; dy \;, \quad x \in \mathbb{R}^n \;,$$

where $B(x,R)$ denotes the ball centered at x with radius R. Show there is a constant C_n, depending only on n such that

$$m\{x \in \mathbb{R}^n :\; Mf(x) > s\} \;\le\; \frac{C_n}{s} \int_{\{x : |f(x)| > s/2\}} |f(y)|\; dy \;.$$

Hint: Consider the function f_s defined by $f_s(x) = |f(x)|$ if $|f(x)| > s/2$, $\;f_s(x) = 0$ otherwise.

Solution: Suppose that $Mf(x) > s$. Then there exists a ball $B(x,R)$ such that

$$\frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)|\; dy \;=\; \frac{1}{|B(x,R)|} \int_{B(x,R)} |f_s(y)|\; dy + \frac{s}{2} \;>\; s \;.$$

We conclude that $\{Mf > s\} \subset \{Mf_s > s/2\}$. The result follow from the Hardy-Littlewood inequality applied to f_s.