Department of Mathematics, University of Michigan
Real Analysis Qualifying Exam
January 7, 2023, 2.00 pm-5.00 pm

Problem 1: Let $E_k,\ k = 1, 2, \ldots, n$, be measurable subsets of $[0, 1]$ such that each point $x \in [0, 1]$ is contained in at least 5 of the sets $E_k,\ k = 1, \ldots, n$. Prove there exists k such that $m(E_k) \geq 5/n$.

Problem 2: For $f \in L^1(\mathbb{R})$ define a sequence of functions $g_n : [0, 1] \to \mathbb{R},\ n = 1, 2, \ldots$, by

$$g_n(x) \ = \ \sum_{k=1}^{n} \frac{1}{\sqrt{k}} f(x + \sqrt{k}) \ , \quad n = 1, 2, \ldots$$

Prove that the sequence $g_n,\ n = 1, 2, \ldots$, is convergent in $L^1([0, 1])$.

Problem 3: Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is absolutely continuous on finite intervals, and $g : [0, 1] \to \mathbb{R}$ an integrable function. Define the function $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \int_0^1 f(x - y)g(y)\ dy$. Show that h is also absolutely continuous on finite intervals.

Problem 4: Let $f_n,\ n = 1, 2, \ldots$, be a sequence of measurable functions on $[0, 1]$ such that $f_n \to 0$ a.e. and $f_n \in L^3([0, 1]),\ n = 1, 2, \ldots$, with $\sup_{n \geq 1} \|f_n\|_3 < \infty$. Prove there exists p with $1 < p < \infty$ such that

$$\lim_{n \to \infty} \int_0^1 f_n(x) g(x)\ dx \ = \ 0 \quad \text{for all } g \in L^p([0, 1]) \ .$$

Problem 5: Let $f(\cdot)$ be an integrable function on $\mathbb{R}^n$ and Mf the corresponding Hardy-Littlewood maximal function

$$Mf(x) \ = \ \sup_{R>0} \frac{1}{|B(x, R)|} \int_{B(x,R)} |f(y)|\ dy \ , \quad x \in \mathbb{R}^n \ ,$$

where $B(x, R)$ denotes the ball centered at x with radius R. Show there is a constant C_n, depending only on n such that

$$m\{x \in \mathbb{R}^n : \ Mf(x) > s\} \ \leq \ \frac{C_n}{s} \int_{\{x:|f(x)|>s/2\}} |f(y)|\ dy \ .$$

Hint: Consider the function f_s defined by $f_s(x) = |f(x)|$ if $|f(x)| > s/2,\ \ f_s(x) = 0$ otherwise.