Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam
January 8, 2023, 2.00 pm-5.00 pm

Problem 1: Use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin \pi x}{x^2 - x} \ .$$

Solution:
For $R > 10$, $0 < \varepsilon < 1$, let $\gamma_{\varepsilon,R} \subset \mathbb{C}$ be the contour made up of the line segments $[-R, -\varepsilon]$, $[\varepsilon, 1-\varepsilon]$, $[1+\varepsilon, R]$ on the real line, together with the semicircles $\{z \in \mathbb{C} : \ |z| = R, \ \Im z > 0\}$, $\{|z| = \varepsilon, \ \Im z > 0\}$, $\{|z-1| = \varepsilon, \ \Im z > 0\}$, and the contour traversed in a counter-clockwise direction. Let

$$I_{\varepsilon,R} = \int_{\gamma_{\varepsilon,R}} \frac{e^{i\pi z}}{z(z-1)} \ dz \ = \ \int_{\gamma_{\varepsilon,R}} f(z) \ dz \ = \ I^1_{\varepsilon,R} + I^2_{\varepsilon} + I^3_{\varepsilon,R} + J^1_{\varepsilon,R} - J^2_{\varepsilon} - J^3_{\varepsilon} \ ,$$

where

$$I^1_{\varepsilon,R} \ = \ \int_{-R}^{-\varepsilon} f(x) \ dx \ , \quad I^2_{\varepsilon} \ = \ \int_{\varepsilon}^{1-\varepsilon} f(x) \ dx \ , \quad I^3_{\varepsilon,R} \ = \ \int_{1+\varepsilon}^{R} f(x) \ dx \ ,$$

$$J^1_{\varepsilon,R} \ = \ i\int_0^{\pi} \frac{e^{-\pi R\sin\theta + i\pi R\cos\theta}}{[Re^{i\theta} - 1]} \ d\theta \ , \quad J^2_{\varepsilon} \ = \ i\int_0^{\pi} \frac{\exp\left[i\pi\varepsilon e^{i\theta}\right]}{\varepsilon e^{i\theta} - 1} \ d\theta \ , J^3_{\varepsilon} \ = \ ie^{i\pi}\int_0^{\pi} \frac{\exp\left[i\pi\varepsilon e^{i\theta}\right]}{\varepsilon e^{i\theta} + 1} \ d\theta \ .$$

By Cauchy's theorem $I_{\varepsilon,R} = 0$. Furthermore one has that

$$\int_{-\infty}^{\infty} \frac{\sin \pi x \ dx}{x^2 - x} \ = \ \lim_{\varepsilon \to 0} \lim_{R \to \infty} \Im\left[I^1_{\varepsilon,R} + I^2_{\varepsilon} + I^3_{\varepsilon,R}\right] \ .$$

We have that

$$|J^1_{\varepsilon,R}| \ \le \ \frac{2\pi}{R} \ , \quad \lim_{\varepsilon \to 0} J^2_{\varepsilon} \ = \ -i\pi, \qquad \lim_{\varepsilon \to 0} J^3_{\varepsilon} \ = \ -i\pi \ .$$

We conclude that the value of the integral is -2π.

Problem 2: Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to the right half plane $\{w \in \mathbb{C} : \Re w > 0\}$ with the property $f(0) = 1$. Prove that

$$|f(z)| \ \le \ \frac{1 + |z|}{1 - |z|} \quad \text{for } z \in \mathbb{D} \ .$$

Solution: We make a conformal transformation g from the right half plane to $\mathbb{D}$ such that $g(1) = 0$. The function $h = g \circ f$ then maps $\mathbb{D}$ to itself and $h(0) = 0$.

By the Schwarz lemma we then have $|h(z)| \le |z|$. The function g is given by the formula

$$g(w) \ = \ \frac{1-w}{1+w} \ , \quad \text{whence } h(z) = \frac{1-f(z)}{1+f(z)} \ \text{ so } f(z) = \frac{1-h(z)}{1+h(z)} \ .$$

Since $|h(z)| \le |z|$ it then follows from the triangle inequality that $|f(z)| \le (1+|z|)/(1-|z|)$.

Problem 3: Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which has the property $f(-z) = -f(z), \ z \in \mathbb{D}$.
a) Show there is a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ such that $g(z^2) = [f(z)]^2, \ z \in \mathbb{D}$.
b) Prove that if $f(\cdot)$ is one to one then the function g in a) is one to one.

Solution: (a) By the anti-symmetry of $f(\cdot)$ the Taylor expansion of $f(\cdot)$ about 0 has the form

$$f(z) \ = \ z\sum_{n=0}^{\infty} a_n z^{2n} \ = \ zh(z^2) \ ,$$

where $h : \mathbb{D} \to \mathbb{C}$ is holomorphic. We can therefore take $g(w) = wh(w)^2$.
(b) Note that since $f(0) = 0$ and $f(\cdot)$ is one to one that $f(z) \ne 0$ for $z \ne 0$. Hence $g(w) \ne 0$ if $w \ne 0$. Suppose now that $w \ne 0$ and $g(w) = a^2 \ne 0$. Letting $w = z^2$ for some $z \ne 0$ we have that $[f(z)]^2 = a^2$ and so $f(z) = a$ or $f(z) = -a$. Since $f(\cdot)$ is one to one and antisymmetric $z = z_a$ or $z = -z_a$ where $f(z_a) = a$ and z_a is unique. In either case $g(w_a) = a^2$ and $w_a = z_a^2$ is unique.

Problem 4: An entire function $f : \mathbb{C} \to \mathbb{C}$ is of exponential type if there exist constants $C_1, C_2 > 0$ such that $|f(z)| \le C_1 e^{C_2|z|}, \ z \in \mathbb{C}$. Show that a function $f(\cdot)$ is of exponential type if and only if its derivative $f'(\cdot)$ is of exponential type.

Solution: Suppose $f(\cdot)$ is of exponential type. By the Cauchy formula

$$f'(z) \ = \ \frac{1}{2\pi i}\int_{|w-z|=1} \frac{f(w)}{(w-z)^2} \ dw \ ,$$

whence

$$|f'(z)| \ \le \ C_1 \sup\{|f(w)| : \ |w-z| = 1\} \ \le \ C_1 e^{C_2} e^{C_2|z|} \ .$$

Conversely if $f'(\cdot)$ is of exponential type we use the representation

$$f(z) - f(0) \ = \ \int_{\gamma_z} f'(w)\, dw \ = \ \int_0^x f'(x')\, dx' + i\int_0^y f'(x+iy')\, dy' \ , \quad \text{where } z = x+iy \ .$$

It follows that

$$|f(z) - f(0)| \ \le \ \int_0^{|x|} C_1 e^{C_2 x'} \ dx' + C_1 e^{C_2|x|}\int_0^{|y|} e^{C_2 y'} \ dy' \ \le \frac{2C_1}{C_2} e^{2C_2|z|} \ .$$

Problem 5: Let $\mathcal{D} \subset \mathbb{C}$ be a domain i.e. open and connected, and $f, g : \mathcal{D} \to \mathbb{C}$ holomorphic functions which have the property that $|f(z)| + |g(z|$ is constant for $z \in \mathcal{D}$. Prove that the functions f and g are constant.
Hint: Apply the maximum principal judiciously.

Solution: Let $z_0 \in \mathbb{D}$ and choose $\theta(z_0) \in [-\pi, \pi]$ such that $|f(z_0) + e^{i\theta(z_0)} g(z_0)| = |f(z_0)| + |g(z_0)|$. Then the function $z \to f(z) + e^{i\theta(z_0)} g(z)$ attains its maximum modulus in $\mathbb{D}$ at z_0, whence the maximum principle implies the function is constant. It then follows from our assumption there exists $K \in \mathbb{C}$ such that

$$|f(z)| + |K - f(z)| \ = \ \text{constant for } z \in \mathbb{D} \ .$$

If $f(\cdot)$ is not constant the open mapping theorem then implies that for some $\varepsilon > 0$ and $K' \in \mathbb{C}$ one has $|w| + |K' - w| = \text{constant}$ for $|w| < \varepsilon$. Since this is clearly false we conclude $f(\cdot)$ is constant-and similarly $g(\cdot)$.