Department of Mathematics, University of Michigan
Analysis Qualifying Exam, May 5, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that for each $z_0 \in \mathbb{C}$ the power series expansion

$$f(z) \ = \ \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

has at least one coefficient a_n which is zero. Show that f is a polynomial.

Solution: For each $z \in \mathbb{C}$ there exists a non-negative integer $n(z) \geq 0$ such that $f^{n(z)}(z) = 0$. Assume $f(\cdot)$ is not a polynomial. Then for every non-negative integer $n \geq 0$ the set $S_n = \{z \in \mathbb{C} : \ f^n(z) = 0\}$ has no accumulation points, and hence is countable. It follows that $\cup_{n=0}^{\infty} S_n$ is also countable. However we know $\cup_{n=0}^{\infty} S_n = \mathbb{C}$, which is uncountable. We conclude that $f(\cdot)$ is a polynomial.

Problem 2: Let U be an open subset of $\mathbb{C}$ and $z_0 \in U$. Suppose that $f(\cdot)$ is a meromorphic function on U with a pole at z_0. Prove there is no holomorphic function $g : U - \{z_0\} \to \mathbb{C}$ such that $f(z) = e^{g(z)}, \ z \in U - \{z_0\}$.

Solution: Since $f(\cdot)$ has a pole at z_0 we have for sufficiently small $r > 0$ that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f'(z)}{f(z} \ dz = \ -N \ ,$$

where $N \neq 0$ is the order of the pole. If $g(\cdot)$ exists then this implies that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} g'(z) \ dz \neq 0 \ .$$

However writing $g(\cdot)$ in its Lorenz expansion centered at z_0 and differentiating we see that the Lorenz expansion of $g'(z)$ has no term in $1/(z - z_0)$. We conclude that

$$\frac{1}{2\pi i} \int_{|z-z_0|=r} g'(z) \ dz = 0 \ , \quad \text{a contradiction.}$$

Problem 3: Use contour integration to evaluate the integral

$$\int_0^{\infty} \frac{x^{-1/3}}{1+x} \ dx \ .$$

Solution: For $\varepsilon, R > 0$ such that $\varepsilon << 1 << R$ we define the contour $\gamma_{\varepsilon,R}$ as follows:

(a) The line segment $\{x+i\varepsilon,\ 0<x<\sqrt{R^2-\varepsilon^2}\}$,
(b) The arc of the circle $\{|z|=R\}$ anti-clockwise from $\sqrt{R^2-\varepsilon^2}+i\varepsilon$ to $\sqrt{R^2-\varepsilon^2}-i\varepsilon$,
(c) The line segment $\{x-i\varepsilon,\ \sqrt{R^2-\varepsilon^2}>x>0\}$,
(d) The arc of the circle $\{|z|=\varepsilon\}$ in the left hand plane.

Letting $f(z)=z^{-1/3}/(1+z)$ we have from the residue theorem that

$$\frac{1}{2\pi i}\int_{\gamma_{\varepsilon,R}} f(z)\,dz \;=\; \mathrm{Res}[f,-1] \;=\; e^{-i\pi/3}\ .$$

Since the integrand $x\to x^{-1/3}/(1+x)$ is integrable in a nbh of 0 and ∞ the limit of the integral of $f(z)$ over the contour (a) as $\varepsilon\to 0,\ R\to\infty$ is

$$I \;=\; \int_0^\infty \frac{x^{-1/3}}{1+x}\,dx\ .$$

For z along the contour (c) one has for $z=x-i\varepsilon$ that $z^{-1/3}\simeq e^{-2\pi i/3}x^{-1/3}$. Hence the limit of the integral of $f(z)$ over the contour (c) as $\varepsilon\to 0,\ R\to\infty$ is $-e^{-2\pi i/3}I$. The integral over (b) is bounded by $CR^{-1/3}$ for a constant C and so goes to zero as $R\to\infty$. The integral over (d) is bounded by $C\varepsilon^{2/3}$ and also converges to zero as $\varepsilon\to 0$. We conclude that

$$[1-e^{-2\pi i/3}]I \;=\; 2\pi i e^{-\pi i/3}\ , \quad \text{whence } I \;=\; \frac{\pi}{\sin\pi/3} \;=\; \frac{2\pi}{\sqrt{3}}\ .$$

Problem 4: Let $\mathbb{D}=\{z\in\mathbb{C}:\ |z|<1\}$ denote the unit disc. Consider a sequence of holomorphic functions $f_n:\mathbb{D}\to\mathbb{C}$ having the Taylor expansions

$$f_n(z)=\sum_{k=1}^{\infty} a_{n,k}z^k$$

where $|a_{n,k}|\le k^{2022}$ for all $k,n\in\mathbb{N}$.
Prove that the sequence $\{f_n\}_{n=1}^\infty$ contains a subsequence uniformly convergent on compact subsets of $\mathbb{D}$.

Solution: Let $K\subset\mathbb{D}$ be compact. Then $\sup\{|z|:\ z\in K\}=r_K<1$. Using the inequality $k\delta<(1+\delta)^k$ for $\delta>0,\ k=1,2,\dots$, we have that for every $z\in K$,

$$|f_n(z)| \;\le\; \sum_{k=1}^\infty k^{2022}r_K^k \;\le\; \frac{1}{\delta^{2022}}\sum_{k=1}^\infty (1+\delta)^{2022k}r_K^k \;=\; \frac{(1+\delta)^{2022}r_K}{\delta^{2022}\left[1-(1+\delta)^{2022}r_K\right]}\ ,$$

provided $\delta>0$ is chosen small enough so $(1+\delta)^{2022}r_K<1$. Hence the family $\mathcal{F}=\{f_n:\mathbb{D}\to\mathbb{C}\}$ is uniformly bounded on any compact subset K of $\mathbb{D}$. The result follows from Montel's theorem.

Problem 5: Find the number of solutions (counted according to multiplicity) in the annulus $\{1 \leq |z| \leq 3\}$ of the equation $z^9 + z^6 + 30z^5 - 3z + 2 = 0$.

Solution: Let $f(z) = z^9 + z^6 + 30z^5 - 3z + 2$ and $g(z) = z^9 + 30z^5$. Then we have that

$$|f(z) - g(z)| \ < \ |g(z)| \quad \text{if } |z| = 1 \text{ or } |z| = 3\ .$$

Hence by the Rouché theorem the number of zeros of $f(\cdot)$ in the annulus is the same as the number of zeros $g(\cdot)$. The zeros of $g(\cdot)$ in the annulus are the roots of $z^4 + 30 = 0$. Since $1 < 30^{1/4} < 3$, the number of zeros of $f(\cdot)$ in the annulus is 4.