Department of Mathematics, University of Michigan
Analysis Qualifying Exam, August 17, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let A be a Lebesgue measurable subset of $[0,1]$ with positive measure. Show there exists $x_1, x_2 \in A$ such that $x_1 - x_2$ is a rational number.

Problem 2: Let $f(\cdot)$ be a locally integrable function on $\mathbb{R}^n$ and Mf the corresponding Hardy-Littlewood maximal function

$$Mf(x) \;=\; \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)| \, dy \;, \quad x \in \mathbb{R}^n \;,$$

where $B(x,R)$ denotes the ball centered at x with radius R.
a) Show that if f is integrable on $\mathbb{R}^n$ then $\sup_{\lambda>0} \lambda \, m\{x \in \mathbb{R}^n : |f(x)| > \lambda\} < \infty$.
b) Let f be the function

$$f(x) \;=\; \begin{cases} 1 & \text{if } |x| < 1; \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Show that Mf is not integrable on $\mathbb{R}^n$, but $\sup_{\lambda>0} \lambda \, m\{x \in \mathbb{R}^n : Mf(x) > \lambda\} < \infty$.

Problem 3: Let $g : [1,\infty) \to \mathbb{R}$ be a non-negative measurable function.
a) Prove the inequality

$$\left(\int_1^\infty g(t) \, dt \right)^3 \;\leq\; \int_1^\infty t^4 g(t)^3 \, dt \;.$$

b) Assuming the integral on the right hand side of the inequality in a) is finite, find all functions g for which the inequality becomes an equality.

Problem 4: Let $f : [0,1] \to \mathbb{R}$ be a continuous function which is absolutely continuous in any interval $[\varepsilon, 1]$ with $0 < \varepsilon < 1$.
a) Is $f(\cdot)$ absolutely continuous on the entire interval $[0,1]$? Prove this or give a counterexample.
b) Suppose now that additionally f is of bounded variation on the entire interval $[0,1]$. In that case is f absolutely continuous on the entire interval $[0,1]$? Prove this or give a counterexample.

Problem 5: Let f and g be bounded measurable functions on $\mathbb{R}^n$. Assume that g is integrable and satisfies $\int g = 0$. For $k > 0$ define the functions g_k and convolution $f * g_k$ by

$$g_k(x) = k^n g(kx), \quad f * g_k(x) \;=\; \int_{\mathbb{R}^n} f(x-y) g_k(y) \, dy \;, \quad x \in \mathbb{R}^n \;.$$

a) Prove that if f is also continuous then $\lim_{k\to\infty} f * g_k(x) = 0$ for almost every $x \in \mathbb{R}^n$.
b) Extend your proof in a) to all bounded measurable functions f. Hint: Use Lusin's theorem.