Department of Mathematics, University of Michigan
Analysis Qualifying Exam, August 17, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Let A be a Lebesgue measurable subset of $[0,1]$ with positive measure. Show there exists $x_1, x_2 \in A$ such that $x_1 - x_2$ is a rational number.

Solution: Enumerate the rational numbers in $[0,1]$ as $r_n,\ n = 1, 2, \ldots,$ and set $A_n = A + r_n$. Then $A_n \subset [0,2]$ and $m(A_n) = m(A) > 0$. Since there are an infinite number of rationals, there exists m, n such that $A_n \cap A_m$ is not empty. If $x \in A_n \cap A_m$ then $x = x_1 + r_n = x_2 + r_m$ where $x_1, x_2 \in A$, whence $x_1 - x_2$ is rational.

Problem 2: Let $f(\cdot)$ be a locally integrable function on $\mathbb{R}^n$ and Mf the corresponding Hardy-Littlewood maximal function

$$Mf(x) \ = \ \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} |f(y)| \ dy \ , \quad x \in \mathbb{R}^n \ ,$$

where $B(x,R)$ denotes the ball centered at x with radius R.
a) Show that if f is integrable on $\mathbb{R}^n$ then $\sup_{\lambda>0} \lambda \, m\{x \in \mathbb{R}^n : |f(x)| > \lambda\} < \infty$.
b) Let f be the function

$$f(x) \ = \ \begin{cases} 1 & \text{if } |x| < 1; \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

Show that Mf is not integrable on $\mathbb{R}^n$, but $\sup_{\lambda>0} \lambda \, m\{x \in \mathbb{R}^n : Mf(x) > \lambda\} < \infty$.

Solution: a) This follows from the Chebyshev inequality

$$\lambda \, m\{x \in \mathbb{R}^n : |f(x)| > \lambda\} \ \le \int_{|f(x)|>\lambda} |f(x)| \ dx \ \le \int_{\mathbb{R}^n} |f(x)| \ dx \ .$$

b) First observe that $Mf(\cdot) \le 1$, whence we can assume $0 < \lambda < 1$. Next there exists a positive integer N such that

$$\frac{1}{2^{(k+N)n}} \ \le \ Mf(x) \ \le \ \frac{1}{2^{(k-N)n}} \quad \text{if } 2^k \le |x| \le 2^{k+1} \ , k = 0, 1, \ldots$$

The inequality implies that $Mf \notin L^1(\mathbb{R}^n)$ but $\sup_{\lambda>0} \lambda \, |\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| < \infty$.

Problem 3: Let $g : [1, \infty) \to \mathbb{R}$ be a non-negative measurable function.
a) Prove the inequality

$$\left(\int_1^\infty g(t)\ dt \right)^3 \ \le \ \int_1^\infty t^4 g(t)^3\ dt\ .$$

b) Assuming the integral on the right hand side of the inequality in a) is finite, find all functions g for which the inequality becomes an equality.

Solution: a) It follows from the Hölder inequality

$$\int fh\ \le \left(\int f^{p'} \right)^{1/p'} \left(\int h^p \right)^{1/p}$$

with $p = 3, p' = 3/2$ that

$$\int_1^\infty g(t)\ dt\ =\ \int_1^\infty t^{-4/3}[t^{4/3}g(t)]\ dt\ \le\ \left(\int_1^\infty t^{-2}\ dt \right)^{2/3} \left(\int_1^\infty t^4 g(t)^3\ dt \right)^{1/3}\ .$$

b) Equality occurs in the Hölder inequality when $h = f^{p'-1} = f^{p'/p}$. This yields $t^{4/3}g(t) = t^{-2/3}$, whence $g(t) = t^{-2}$.

Problem 4: Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which is absolutely continuous in any interval $[\varepsilon, 1]$ with $0 < \varepsilon < 1$.
a) Is $f(\cdot)$ absolutely continuous on the entire interval $[0, 1]$? Prove this or give a counterexample.
b) Suppose now that additionally f is of bounded variation on the entire interval $[0, 1]$. In that case is f absolutely continuous on the entire interval $[0, 1]$? Prove this or give a counterexample.

Solution: a) No. An example is $f(x) = x \sin(1/x)$. The function $f(\cdot)$ is C^1 on any interval $[\varepsilon, 1]$ but

$$\int_0^1 |f'(x)|\ dx\ =\ \int_0^1 |\sin(1/x) - x^{-1}\sin(1/x)|\ dx\ =\ \infty\ .$$

b) Yes, since

$$\int_0^1 |f'(x)|\ dx\ \le TV_{[0,1]}(f)\ .$$

Problem 5: Let f and g be bounded measurable functions on $\mathbb{R}^n$. Assume that g is integrable and satisfies $\int g = 0$. For $k > 0$ define the functions g_k and convolution $f * g_k$ by

$$g_k(x) = k^n g(kx), \quad f * g_k(x)\ =\ \int_{\mathbb{R}^n} f(x - y) g_k(y)\ dy\ , \quad x \in \mathbb{R}^n\ .$$

a) Prove that if f is also continuous then $\lim_{k\to\infty} f * g_k(x) = 0$ for almost every $x \in \mathbb{R}^n$.

b) Extend your proof in a) to all bounded measurable functions f. Hint: Use Lusin's theorem.

Solution: a) Since the integral of g is zero, we have that

$$f * g_k(x) \ = \ \int_{\mathbb{R}^n} f(x - z/k)g(z) \ dz \ = \ \int_{\mathbb{R}^n} [f(x - z/k) - f(x)]g(z) \ dz \ .$$

Hence we have that

$$|f * g_k(x)| \ \le \ \sup_{|z|<\sqrt{k}} |f(x - z/k) - f(x)| \int_{|z|<\sqrt{k}} |g(z)| \ dz + 2\|f\|_\infty \int_{|z|>\sqrt{k}} |g(z)| \ dz \ .$$

The first term on the RHS converges to 0 as $k \to \infty$ since f is continuous at x and g is integrable. The second term converges to zero since g is integrable.
b) By Lusin's theorem there exists for any $\varepsilon > 0$ a continuous function f_ε of compact support such that

$$m(A_\varepsilon) \ = \ m\{x : \ |x| < 2R, \ f(x) \ne f_\varepsilon(x)\} \ < \ \varepsilon \ , \quad \|f_\varepsilon\|_\infty \le \|f\|_\infty \ .$$

We write

$$|f * g_k(x) - f_\varepsilon * g_k(x)| \ \le \ 2\|f\|_\infty \left[\int_{|y|>M} |g(y)| \ dy + k^n \|g\|_\infty m\{y : x - y \in A_\varepsilon, \ |y| < M/k\} \right] \ .$$

At this point both ε, M are fixed with ε small and M large. Now we let $k \to \infty$ in the second term on the RHS of the inequality. We have for $\delta > 0$ small and $0 < \gamma < 1$ that

$$m\{x : |x| < R \text{ and } m\{y : x - y \in A_\varepsilon, \ |y| < \delta\} > \gamma\delta^n\} \ \le \ \frac{C\varepsilon}{\gamma} \ ,$$

for some constant C. This follows from the Chebyshev inequality applied to the convolution function

$$f(x) \ = \ \frac{1}{\delta^n} \int \mathbb{1}_A(x - y)\mathbb{1}_{B_\delta}(y) \ dy \ .$$

We apply this to the previous inequality with $\delta = M/k$ and $\gamma = 1/M^{n+1}$. Then outside a set of measure $CM^{n+1}\varepsilon$ the second term on the RHS is bounded by $1/M$ as $k \to \infty$. Finally first let $\varepsilon \to 0$ and then $M \to \infty$ to get the result.