Department of Mathematics, University of Michigan
Analysis Qualifying Exam, August 18, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{\cos x\ dx}{(1+x^2)^2} \ .$$

Problem 2: Find a conformal mapping from the quarter disc

$$\{z \in \mathbb{D}: \ z = re^{i\theta}, \ r \in (0,1), \ \theta \in (0, \pi/2)\}$$

to the infinite strip

$$\{z \in \mathbb{C}: \ z = x + iy, \ x \in \mathbb{R}, \ y \in (0,1)\} \ .$$

You may write your solution as a composite of simpler maps.

Problem 3: Suppose $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function on the unit disk $\mathbb{D}$ which satisfies $|f(z)| \le 3$ for all $|z| < 1$, and $f(1/2) = 2$.
a) Show that $f(0) \neq 0$.
b) Extend your result in a) by showing that $f(\cdot)$ has no zeros in the disk $|z| < 1/8$.

Problem 4: Consider a function $f(z)$ that is analytic for $z \neq 0$ and such that there exists a sequence $z_j, \ j = 1, 2, \ldots$, such that $f(z_j) = 0, \ j \ge 1$, and $\lim_{j\to\infty} z_j = 0$.
a) Prove that f cannot have a pole at $z = 0$.
b) Show by explicit example that there does exist such f which has an essential singularity at $z = 0$.

Problem 5: Let U be a bounded connected domain in $\mathbb{C}$ and $f : U \to U$ a holomorphic function which satisfies $f(z_0) = z_0$ and $|f'(z_0)| < 1$ for some $z_0 \in U$. For $n = 1, 2, \ldots$, let $f^{(n)}$ be the composition function defined inductively by $f^{(1)} = f, \ f^{(n+1)} = f^{(n)} \circ f$. Prove that $f^{(n)}$ converges uniformly to z_0 on compact subsets of U.