Department of Mathematics, University of Michigan
Analysis Qualifying Exam, August 18, 2022
Morning Session, 9.00 AM-12.00

Problem 1: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{\cos x \ dx}{(1+x^2)^2} \ .$$

Solution: We use the calculus of residues. Let γ_R be the contour consisting of the line segment $[-R, R]$ on the real axis combined with the semi-circle of radius R in the upper half plane. The direction of the contour is counter clockwise. Thus

$$\frac{1}{2\pi i}\int_{\gamma_R} \frac{e^{iz} \ dz}{(1+z^2)^2} \ = \ \mathrm{Res}(f(\cdot), i) \ , \quad \text{where } f(z) = \frac{e^{iz}}{(1+z^2)^2} \ .$$

We have that

$$f(z) = \frac{e^{iz}}{(z-i)^2(z+i)^2} \ , \quad e^{iz} = e^{-1} + ie^{-1}(z-i) + O[(z-i)^2] \ ,$$

$$\frac{1}{(z+i)^2} \ = \ -\frac{1}{4} - \frac{i}{4}(z-i) + O[(z-i)^2] \ ,$$

whence we conclude that

$$f(z) \ = \ \frac{1}{(z-i)^2}\left[-\frac{e^{-1}}{4} - \frac{ie^{-1}}{2}(z-i) + O[(z-i)^2]\right] \ .$$

Hence $\mathrm{Res}(f(\cdot), i) = -ie^{-1}/2$. We observe that

$$\lim_{R\to\infty}\int_{\gamma_R\cap\{\Im z=0\}} f(z) \ dz \ = \ 2\int_0^\infty \frac{\cos x \ dx}{(1+x^2)^2} \ .$$

If we show that

$$\lim_{R\to\infty}\int_{\gamma_R\cap\{|z|=R\}} f(z) \ dz \ = \ 0 \ ,$$

then we have from the residue theorem that the value of the integral in the problem is $\pi e^{-1}/2$. Now

$$|f(z)| \ \le \ \frac{1}{(R-1)^4}, \quad \text{for } |z| = R, \ \Im z > 0, \quad \int_{\gamma_R\cap\{|z|=R\}} |f(z)| \ |dz| \ \le \ \frac{\pi R}{(R-1)^4} \ .$$

Letting $R \to \infty$ we conclude the integral of $f(\cdot)$ on the semi-circle converges to 0 as $R \to \infty$.

Problem 2: Find a conformal mapping from the quarter disc

$$\{z \in \mathbb{D} : \ z = re^{i\theta}, \ r \in (0,1), \ \theta \in (0, \pi/2)\}$$

to the infinite strip

$$\{z \in \mathbb{C} : \ z = x + iy, \ x \in \mathbb{R}, \ y \in (0,1)\} \ .$$

You may write your solution as a composite of simpler maps.

Solution: If $f_1(z) = z^2$ then f_1 maps the quarter disc $\mathcal{D}$ to the half disk $\mathcal{D}_1 = \{z : |z| < 1, \ \Im z > 0\}$. We can map $\mathcal{D}_1$ to a quadrant using a FLT by sending -1 to 0 and $+1$ to ∞. Thus we take $f_2(z) = (1+z)/(1-z)$, which maps $\mathcal{D}_1$ to $\mathcal{D}_2 = \{z = re^{i\theta}, \ 0 < \theta < \pi/2\}$. Next $f_3(z) = \frac{2}{\pi} \log z$ maps $\mathcal{D}_2$ to the infinite strip $0 < \Im z < 1$. The conformal mapping is therefore $f = f_3 \circ f_2 \circ f_1$.

Problem 3: Suppose $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function on the unit disk $\mathbb{D}$ which satisfies $|f(z)| \le 3$ for all $|z| < 1$, and $f(1/2) = 2$.
a) Show that $f(0) \neq 0$.
b) Extend your result in a) by showing that $f(\cdot)$ has no zeros in the disk $|z| < 1/8$.

Solution: a). We wish to use the Schwarz lemma, whence we define $f_1(z) = f(z)/3$, which maps the unit disk $\mathbb{D} = \{z : |z| < 1\}$ to itself. Then $f_1(1/2) = 2/3$. Next we use conformal mappings on $\mathbb{D}$ to construct a function $g : \mathbb{D} \to \mathbb{D}$ from f with $g(0) = 0$. Hence we need FLTs, which are conformal mappings of $\mathbb{D}$, such that $0 \to 1/2$ and $2/3 \to 0$. The relevant mappings are

$$z \to h(z) = \frac{z + 1/2}{1 + z/2} \ , \quad w \to k(w) = \frac{w - 2/3}{1 - 2w/3} \ .$$

Then $g = k \circ f_1 \circ h$. The Schwarz lemma implies that $|g(z)| < |z|, \ z \in \mathbb{D} - \{0\}$. Note that $h(-1/2) = 0, \ k(0) = -2/3$. If $f(0) = 0$ then $f_1(0) = 0$ and so $g(-1/2) = -2/3$, contradicting Schwarz. We conclude that $f(0) \neq 0$.
b) We may extend the argument to the disk by observing that h^{-1} takes the circle centered at 0 with radius r to the circle with equation $h(z)\bar{h}(z) = r^2$, which is given by

$$\left(1 - \frac{r^2}{4}\right)(x^2 + y^2) + (1 - r^2)x + \frac{1}{4} - r^2 \ = 0 \quad z = x + iy \ .$$

This circle has center $[-2(1-r^2)/(4-r^2), 0]$ and radius R satisfying

$$R^2 \ = \ \frac{4(1-r^2)^2}{(4-r^2)^2} - \frac{1-4r^2}{4-r^2} \ .$$

The result follows since

$$\frac{2(1-r^2)}{(4-r^2)} + R \ < \frac{2}{3} \quad \text{when } r = \frac{1}{8} \ .$$

Problem 4: Consider a function $f(z)$ that is analytic for $z \neq 0$ and such that there exists a sequence $z_j, \ j = 1, 2, \ldots,$ such that $f(z_j) = 0, \ j \ge 1$, and $\lim_{j\to\infty} z_j = 0$.

a) Prove that f cannot have a pole at $z = 0$.
b) Show by explicit example that there does exist such f which has an essential singularity at $z = 0$.

Solution: a) If f has a pole at $z = 0$ then $f(z) = z^{-N}g(z)$ where $N \geq 1$ is an integer and g is analytic on $\mathbb{C}$ with $g(0) \neq 0$. Since g is continuous it follows that $f(z_j) \neq 0$ for j sufficiently large, a contradiction.

b) An example is

$$f(z) \quad = \quad \exp\left[\frac{1}{z}\right] - 1 \ , \qquad z_j = \frac{1}{2\pi j i} \ , \ j = 1, 2, \ldots$$

Problem 5: Let U be a bounded connected domain in $\mathbb{C}$ and $f : U \to U$ a holomorphic function which satisfies $f(z_0) = z_0$ and $|f'(z_0)| < 1$ for some $z_0 \in U$. For $n = 1, 2, \ldots$, let $f^{(n)}$ be the composition function defined inductively by $f^{(1)} = f$, $f^{(n+1)} = f^{(n)} \circ f$. Prove that $f^{(n)}$ converges uniformly to z_0 on compact subsets of U.

Solution: Since $|f'(z_0)| < 1$ and f' is continuous there exists $r, \delta > 0$ such that $|f'(z)| \leq 1 - \delta$ if $|z - z_0| \leq r$. It follows that $|f(z) - f(z_0)| \leq (1 - \delta)|z - z_0|$ if $z \in D(z_0, r) = \{z : \ |z - z_0| < r\}$. Since $f(z_0) = z_0$ we conclude that $f(D(z_0, r)) \subset D(z_0, (1 - \delta)r)$. Proceeding by induction we have further that $f^{(n)}(D(z_0, r)) \subset D(z_0, (1 - \delta)^n r)$, $n = 1, 2, \ldots$. Letting $n \to \infty$, it follows that $f^{(n)}$ converges uniformly to z_0 on compact subsets of $D(z_0, r)$.

We extend the result to compact subsets of U by using Montel's theorem. Since U is bounded the family of holomorphic functions $f^{(n)}$, $n \geq 1$, is bounded on every compact subset of U. Suppose the sequence $f^{(n)}$, $n \geq 1$, does not converge uniformly to z_0 on a compact subset $K \subset U$. Then there exists a sequence of points $z_j \in K$, $j = 1, 2, \ldots$, and a subsequence $f^{(n_j)}$, $j = 1, 2, \ldots$, of the family $f^{(n)}$, $n \geq 1$, such that $|f^{(n_j)}(z_j) - z_0| \geq \delta > 0$ for some positive δ. By Montel's theorem there exists a subsequence $f^{(m_k)}$, $k = 1, 2, \ldots$, of $f^{(n_j)}$, $j = 1, 2, \ldots$, which converges uniformly on all compact subsets of U to a holomorphic function $f^{(\infty)}$. Since the sequence z_j, $j \geq 1$, lies in the compact set K there exists a subsequence which has a limit point $z_\infty \in K$. We claim that $|f^{(\infty)}(z_\infty) - z_0| \geq \delta$. This follows from the fact that the derivatives of $f^{(n)}$ are uniformly bounded in a nbh of z_∞, which is a consequence of the Cauchy integral formula. Since we have shown that $f^{(\infty)} \equiv z_0$ in $D(z_0, r)$ it follows by analytic continuation that $f^{(\infty)} \equiv z_0$ in U, but this contradicts the inequality $|f^{(\infty)}(z_\infty) - z_0| \geq \delta$.