Complex Analysis Qualifying Review

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1. Let D be the ellipse $\frac{x^2}{1} + \frac{y^2}{4} < \frac{1}{2021}$ in the complex plane (so $z = x + iy$), and let $z_1, z_2 \in D$ be two distinct points. Let $\varphi\colon D \to D$ be an analytic (=holomorphic) map such that $\varphi(z_1) = z_1$ and $\varphi(z_2) = z_2$. Prove that φ is the identity map.

Since $D \neq \mathbb{C}$ is convex, and hence simply connected, we can pick a conformal map $h\colon D \to \mathbb{D}$ with $h(z_1) = 0$. Set $\zeta := h(z_2)$ and $\psi := h \circ \varphi \circ h^{-1}$. Then $\psi\colon \mathbb{D} \to \mathbb{D}$ is analytic, and $\psi(0) = 0$, $\psi(\zeta) = \zeta$. By the Schwartz Lemma there exists $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ such that $\psi(z) = \lambda z$. But $\psi(\zeta) = \zeta$ now gives $\lambda = 1$, so $\psi = \mathrm{id}$, yielding $\varphi = h^{-1} \circ \psi \circ h = h^{-1} \circ h = \mathrm{id}$.

2. Let $D = \mathbb{D}(0, r)$ be a disc centered at the origin, and let f be a function defined and analytic in a neighborhood of the closure of D. Assume that $|f(z)| < r^2$ when $|z| = r$. Prove that there exists $\varepsilon > 0$ such that if $|\zeta| \leq \varepsilon$, then the equation $f(z) = z^2 + \zeta$ has exactly two solutions (with multiplicity) in D.

The function $h(z) = z^2 - \zeta$ has two zeros in D as long as $|\zeta| \leq \varepsilon < r^2$. Moreover, we have $\delta := \inf_{|z|=r}(|z|^2 - |f(z)|) > 0$, so if we pick $\epsilon < \delta$, then $|-f(z)| < |z^2 + \zeta|$ when $|z| = r$. Rouché's theorem now shows that the function $z^2 - \zeta - f(z)$ has exactly two zeros in D.

3. Let a and b be complex numbers with $0 < |a| < |b|$. Find three different Laurent series expansions of the rational function $f(z) = \frac{1}{(z-a)(z-b)}$, valid in three different regions, each of which is invariant under rotation around the origin.

We have $f(z) = \frac{1}{a-b} g(z)$, where $g(z) = \frac{1}{z-a} - \frac{1}{z-b}$.

On the disc $\{|z| < |a|\}$ we have

$$\frac{1}{z-a} = -\frac{1}{a}\frac{1}{1 - z/a} = -\frac{1}{a}\sum_0^\infty a^{-n} z^n = -\sum_0^\infty a^{-n-1} z^n.$$

and similarly $\frac{1}{z-b} = -\sum_0^\infty b^{-n-1} z^n$, so that

$$f(z) = \sum_{n=0}^\infty \frac{b^{-n-1} - a^{-n-1}}{a-b} z^n.$$

On the disc $\{|a| < |z| < |b|$ we have

$$\frac{1}{z-a} = \frac{1}{z}\frac{1}{1-\frac{a}{z}} = \sum_{-\infty}^{-1} a^{-n-1} z^n$$

whereas $\frac{1}{z-b} = -\sum_0^\infty b^{-n-1} z^n$ as before, so that

$$f(z) = \sum_{-\infty}^{-1} \frac{a^{-n-1}}{a-b} z^n + \sum_{n=0}^{\infty} \frac{b^{-n-1}}{a-b} z^n.$$

Finally, on $\{|z| > |b|\}$, we have $\frac{1}{z-a} = \sum_{-\infty}^{-1} a^{-n-1} z^n$ and $\frac{1}{z-b} = \sum_{-\infty}^{-1} b^{-n-1} z^n$, so

$$f(z) = \sum_{-\infty}^{-1} \frac{a^{-n-1} - b^{-n-1}}{a-b} z^n.$$

4. Let $\mathbb{H} = \{\operatorname{Re} z > 0\}$ be the right half plane, and f an analytic function on $\mathbb{H}$. Assume that $|f(z)| \leq \frac{1}{(\operatorname{Re} z)^2}$ for all $z \in \mathbb{H}$. Prove that $|f'(1)| \leq \frac{27}{4}$.

We use the Cauchy estimates on the disc $\overline{\mathbb{D}}(1, r)$, where $0 < r < 1$. On $\partial\mathbb{D}(1, r)$ we have $|f(z)| \leq \frac{1}{(1-r)^2}$, so the Cauchy estimate gives

$$|f'(1)| \leq \frac{1}{(1-r)^2 r}.$$

By calculus, the function $h(r) = r(1-r)^2$ attains its maximum on $(0, 1)$ at $r = 1/3$, and $h(1/3) = 4/27$. Thus $|f'(1)| \leq 27/4$.

5. Let $D \subset \mathbb{C}$ be a domain (i.e. a connected open set) and $(g_n)_n$ a sequence of uniformly bounded analytic functions on D. Assume that there exists a point $\zeta \in D$ such that for all $m \geq 0$, the derivatives $g_n^{(m)}(\zeta)$ converge to zero as $n \to \infty$. Prove that $(g_n)_n$ converges locally uniformly on D (i.e. uniformly on each compact subset of D) to 0 as $n \to \infty$.

We argue by contradiction, so suppose the statement is false. Then there exist a subsequence $(n_j)_j$, a compact set $K \subset D$, and $\varepsilon > 0$, such that $\sup_K |g_{n_j}| \geq \varepsilon$ for all j. By Montel's theorem, we may, after passing to a further subsequence, assume that $(g_{n_j})_j$ converges locally uniformly to some analytic function g on D. By the Cauchy estimates, the derivatives $g_{n_j}^{(m)}$ also converge locally uniformly to $g^{(m)}$ as $j \to \infty$. By the assumption, this implies that $g^{(m)}(\zeta) = 0$ for all m, which implies that $g \equiv 0$, since D is connected. As g_{n_j} converges uniformly to g on K, we get $\lim_j \sup_K |g_{n_j}| = 0$, a contradiction.