# Real Analysis Qualifying Review

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Notation: $m^*(E)$ is the Lebesgue outer measure of $E$, $m(E)$ is the Lebesgue measure of $E$.

**1.** Let $A_1 \subset A_2 \subset \mathbb{R}^n$. Assume that $A_1$ is Lebesgue measurable and $m(A_1) = m^*(A_2) < \infty$. Show that $A_2$ is also Lebesgue measurable.

**2.** Let $f$ be a measurable function on $E \subset \mathbb{R}$. Assume that

$$\int_E |x|^{1/4} |f(x)|^2 \, dx < \infty, \qquad \int_E x^4 |f(x)|^3 \, dx < \infty.$$

Is $f \in L^1(E)$? Justify your assertion.

**3.** (a) Construct a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in L^p(\mathbb{R})$ for all $0 < p < \infty$, but $f \notin L^\infty(\mathbb{R})$.

(b) Let $(X, \mathcal{A}, \mu)$ be a measure space such that for any $t > 0$ there exists a set $E_t \in \mathcal{A}$, with $\mu(E_t) = t$. Construct a function $f : X \to \mathbb{R}$ such that $f \in L^p(\mu)$ for all $0 < p < \infty$, but $f \notin L^\infty(\mu)$.

**4.** Let $f, g$ be nonnegative measurable functions on $E \subset \mathbb{R}^n$, and assume $fg \in L^1(E)$. Let $E_y = \{x \in E \,|\, g(x) \geq y\}$. Show that

a) for every $y > 0$,

$$F(y) := \int_{E_y} f(x) \, dx < \infty.$$

b) Is $F \in L^1(0, +\infty)$? Justify your answer.

**5.** Let $f$ be a function defined on $\mathbb{R}^n$. Assume that for any $\epsilon > 0$, there are $g, h \in L^1(\mathbb{R}^n)$, satisfying

$$g(x) \leq f(x) \leq h(x), \qquad a.e. \ x \in \mathbb{R}^n,$$

and

$$\int_{\mathbb{R}^n} \big(h(x) - g(x)\big) \, dx < \epsilon.$$

Show that $f \in L^1(\mathbb{R}^n)$.