Complex Analysis Qualifying Review

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The terms "holomorphic" and "analytic" are synomymous. The complex plane is denoted $\mathbb{C}$, and $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ is the unit disc.

1. Let $g_n(z)$, $n = 1, 2, \ldots$ be a sequence of entire functions having only real zeros. Suppose that $g_n(z)$ converges locally uniformly (i.e. uniformly on compact subsets) on $\mathbb{C}$ to an entire function $g(z)$, and that $g(z)$ is not identically zero. Prove that $g(z)$ only have real zeros.

2. For which integers $k \geq 1$ does there exist a holomorphic function $f(z)$ defined near the origin, such that $f(\frac{1}{n}) = f(-\frac{1}{n}) = \frac{1}{n^k}$ for infinitely many integers $n \geq 1$.

3. Construct a conformal mapping f of the domain

$$\Omega = \{z \in \mathbb{C} \mid |z| < 1, |z - \tfrac{i}{2}| > \tfrac{1}{2}\}$$

onto the unit disc $\mathbb{D} = \{w \in \mathbb{C} \mid |w| < 1\}$ with $f(-\frac{i}{3}) = 0$. You may express f as a composition of simpler maps. Draw figures to illustrate each step of the construction.

4. Let $\varphi(z)$ be a holomorphic function on the unit disc $\mathbb{D}$, and set $f(z) = z + z^2\varphi(z)$. Assume that one of the following conditions holds:

(a) $f(\mathbb{D}) \subset \mathbb{D}$;
(b) f is one-to-one on $\mathbb{D}$ and $f(\mathbb{D}) \supset \mathbb{D}$.

Prove that $\varphi(z) = 0$ for all $z \in \mathbb{D}$.

5. Let $f(z)$ be an entire function. Assume that f takes real values on the real axis, and purely imaginary values on the line $\operatorname{Re} z = \operatorname{Im} z$ (i.e. $y = x$, where $z = x + iy$). Prove that f takes real values on the imaginary axis. Also give an example of a function satisfying the hypotheses.