Department of Mathematics, University of Michigan
Complex Analysis Qualifying Exam, August 19, 2021
Morning Session, 9.00 AM-12.00

Problem 1: Let $\mathbb{D} = \{|z| < 1\}$ be the unit disk in $\mathbb{C}$ and $f : \mathbb{D} \to \mathbb{C}$ an analytic function, which extends to a continuous function on the closure $\bar{\mathbb{D}}$ of $\mathbb{D}$.

(a) Show that for any $z_0 \in \mathbb{D}$ there is a constant C, independent of $f(\cdot)$, such that the derivative $f'(\cdot)$ satisfies $|f'(z_0)| \le C \sup_{|z|=1} |f(z)|$.

(b) Give an example to show that C may not be chosen independent of $z_0 \in \mathbb{D}$.

Problem 2: Use contour integration to find the value of the integral

$$\int_0^\infty \frac{dx}{1 + x^{2021}}.$$

Problem 3: Let $z_n,\ n = 1, 2, \ldots,$ be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} |z_n| = \infty$ and $a_n,\ n = 1, 2, \ldots,$ a sequence in $\mathbb{C} - \{0\}$ satisfying

$$\sum_{n=1}^\infty \frac{|a_n|}{|z_n|} < \infty .$$

(a) Show that the function $f(\cdot)$ defined by

$$f(z) = \sum_{n=1}^\infty \frac{a_n}{z - z_n} , \quad z \in \mathbb{C} ,$$

is a meromorphic function $f : \mathbb{C} \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with simple poles at the points $z_n,\ n = 1, 2, \ldots$.

(b) Does $f(\cdot)$ extend to a meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$? Explain your answer.

Problem 4: Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(1) = 1$ and $|f(z)| = 1$ if $|z| = 1$.

(a) Show that

$$\overline{f(z)} f\left(\frac{1}{\bar{z}}\right) = 1 \quad \text{for } z \in \mathbb{C} .$$

(b) Conclude that $f(z) = z^n$ for some integer $n \ge 0$.

Problem 5:

(a) Show that $\mathbb{D}$ is not conformally equivalent to $\mathbb{C}$.

(b) Find an analytic mapping $f : \mathbb{D} \to \mathbb{C}$ such that $f(\mathbb{D}) = \mathbb{C}$.