Department of Mathematics, University of Michigan
Analysis Qualifying Exam, May 14, 2020
Morning Session, 9.00 AM-12.00

Problem 1: Suppose $f(\cdot)$ is holomorphic on the punctured unit disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and satisfies the inequality

$$|f(z)| \ \le \ \log \frac{1}{|z|} \ , \quad 0 < |z| < 1 \ .$$

Show that $f(\cdot) \equiv 0$.

Problem 2: Use contour integration to evaluate the integral

$$\int_0^\infty \frac{x^\alpha}{(x+2)^2} \, dx \quad \text{for } -1 < \alpha < 1 \ .$$

Sketch the contour you use and show all estimates.

Problem 3: Find the Laurent series expansion about the origin of the function

$$f(z) \ = \ \frac{z^2+3z+5}{2z^2-5z-3}$$

which converges in the annulus $1 < |z| < 2$.

Problem 4: Is there a conformal mapping from $\mathbb{C}\backslash\{0\}$ onto the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$? Either prove no such mapping exists or exhibit one.

Problem 5: Let $f : \mathbb{C} \to \mathbb{C}$ be the function $f(z) = z^3 - \exp[z^3 - 4] + 1$.

(a) Find how many solutions counted according to multiplicity there are to the equation $f(z) = 0$ in the disk $|z| < 3/2$.

(b) Find the number of distinct solutions to $f(z) = 0$ in $|z| < 3/2$.

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Afternoon Session, 2.00-5.00 PM

Problem 1: Let $E \subset (0,1)$ be a measurable set such that for any interval $(a,b) \subset (0,1)$, there exists an interval $(c,d) \subset (a,b) \setminus E$ with

$$d - c \geq \frac{a}{10}(b-a).$$

Prove that $m(E) = 0$.

Problem 2: Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function such that

$$f(y) \ \leq \ f(x) + (x^2+y^2)(x-y) \quad \text{for } -\infty < y < x < \infty \ .$$

Show that the derivative function $x \to f'(x)$ exists a.e. on $\mathbb{R}$.

Problem 3: Let $r_n, \ n = 1, 2, \ldots,$ be an enumeration of the rationals in the interval $[0,1]$ and consider the function $f : [0,1] \to \mathbb{R} \cup \infty$ defined by

$$f(x) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{|x - r_n|^{1/3}} \ , \quad 0 \leq x \leq 1 \ .$$

Show that $f \in L^2(0,1)$.

Problem 4: Let $f_n, \ n = 1, 2, ...,$ be the sequence of functions on $(0, \infty)$ defined by

$$f_n(x) \ = \ \frac{1}{n}\left(1 - \frac{x}{n}\right)^n e^x \ , \quad 0 < x < n, \quad f_n(x) = 0, \quad x \geq n.$$

Prove that the sequence $a_n, \ n = 1, 2, \ldots,$ given by

$$a_n \ = \ \int_0^\infty f_n(x) \ dx \quad \text{converges and identify } a_\infty = \lim_{n\to\infty} a_n.$$

Problem 5: Suppose f is a continuously differentiable function on $\mathbb{R}$ satisfying $f(0) = 0, \ |f(x)| \leq |x|^{-1/2}, \ x \neq 0$. Let g be in $L^1(\mathbb{R})$.

(a) Show there is a constant C such that $m\{|g| > \alpha\} \leq C/\alpha$ for all $\alpha > 0$.

(b) Show that the function $h(x) = f(g(x))$ is in $L^1(\mathbb{R})$.